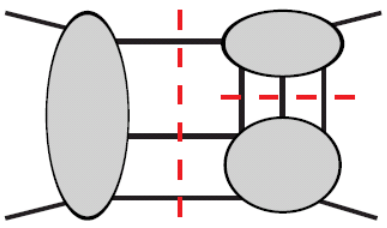
Gravity as a Double Copy of Gauge Theory and Implications for UV Properties

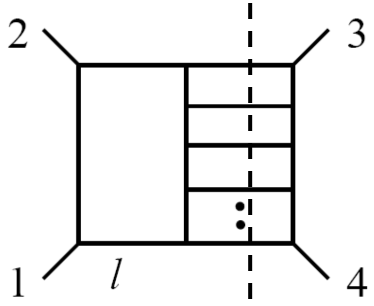
Turin ERC Workshop

March 9, 2012

Zvi Bern, UCLA & CERN

Based on various papers with John Joseph Carrasco, Scott Davies, Tristan Dennen, Lance Dixon, Yu-tin Huang, Harald Ita, Henrik Johansson and Radu Roiban.





Outline

1) **Fantastic progress in computing scattering amplitudes.**

2) **A hidden structure in gauge and gravity amplitudes**

   — **a duality between color and kinematics.**

   — **gravity as a double copy of gauge theory.**

3) **Application: Reexamination of UV divergences of $N = 8$ supergravity (update)**

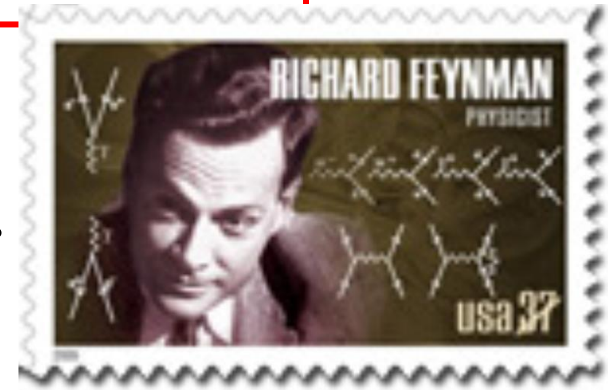
   — **Lightning review of UV properties.**

   — **Four loops in $N = 8$ supergravity.**

   — **A three-loop surprise in $N = 4$ supergravity.**

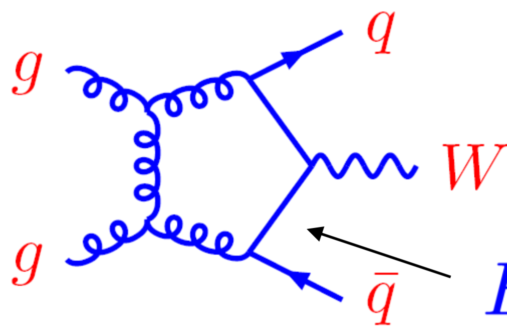
   — **Consequences and prospects for future.**

Why are Feynman diagrams difficult for high-loop or high-multiplicity processes?



Vertices and propagators involve unphysical gauge-dependent off-shell states. An important origin of the complexity.

$$\int \frac{d^3\vec{p}\, dE}{(2\pi)^4}$$



Individual Feynman diagrams unphysical

$$E^2 - \vec{p}^{\,2} \neq m^2$$

Einstein's relation between momentum and energy violated in the loops. Unphysical states! Not gauge invariant.

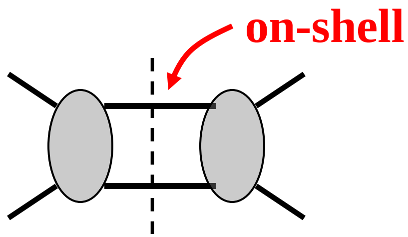
- **All steps should be in terms of gauge invariant on-shell physical states. On-shell formalism. Need to rewrite quantum field theory!**

ZB, Dixon, Dunbar, Kosower (BDDK)

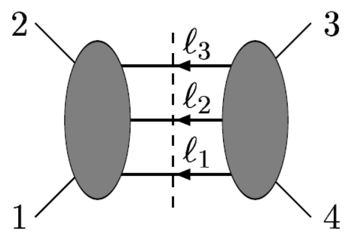
Bern, Dixon, Dunbar and Kosower (1998)

Unitarity Method: Rewrite of QFT

Two-particle cut:

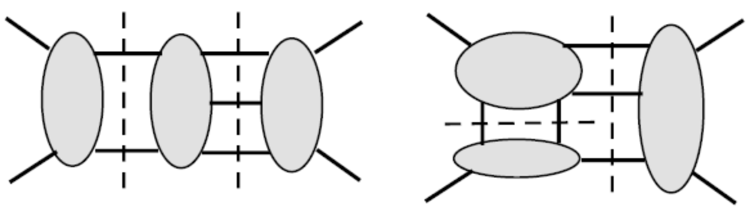


Three-particle cut:



Systematic assembly of complete amplitudes from cuts for any number of particles or loops.

Generalized unitarity as a practical tool:



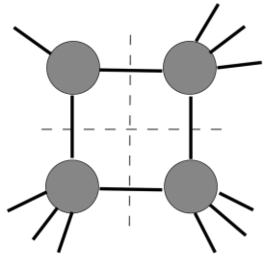
Bern, Dixon and Kosower
Britto, Cachazo and Feng; Forde;
Ossala, Pittau, Papadopolous, and many others

Different cuts merged to give an expression with correct cuts in all channels.

Now a standard tool

complex momenta to solve cuts

Britto, Cachazo and Feng



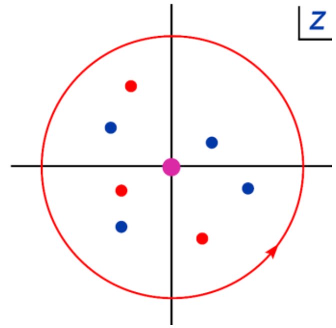
On-Shell Recursion for Tree Amplitudes

Britto, Cachazo, Feng and Witten

Consider amplitude under complex shifts of the momenta.

$$p_1^\mu(z) = p_1^\mu - zq^\mu \quad p_n^\mu(z) = p_n^\mu + zq^\mu \qquad q^2 = 0, \; p \cdot q = 0$$

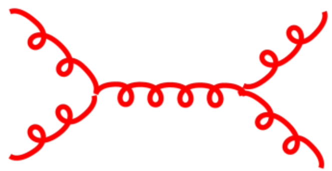
$$(p_i^\mu(z))^2 = 0 \qquad \text{complex momenta}$$



If $A(z) \to 0, \quad z \to \infty$

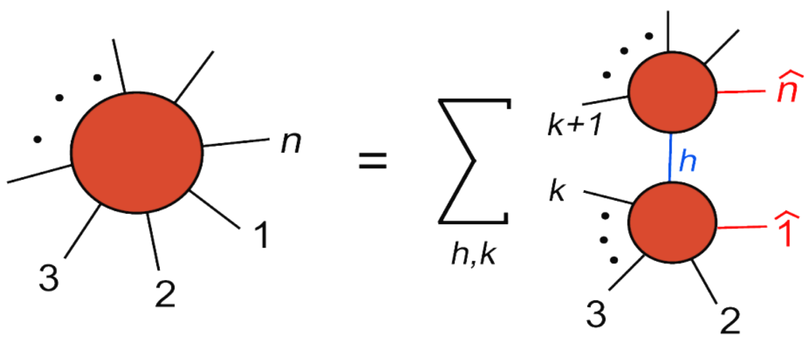
A(z) is amplitude with shifted momenta

$$\oint_{C_\infty} \frac{A(z)}{z} dz = 0 \quad \Rightarrow A(z=0) = -\sum_\alpha \text{Res}_\alpha \frac{A(z)}{z}$$



$$A(z) = \sum_\alpha \frac{c_\alpha}{z - z_\alpha}$$

on-shell amplitude



Sum over residues gives the on-shell recursion relation

Poles in *z* come from kinematic poles in amplitude.

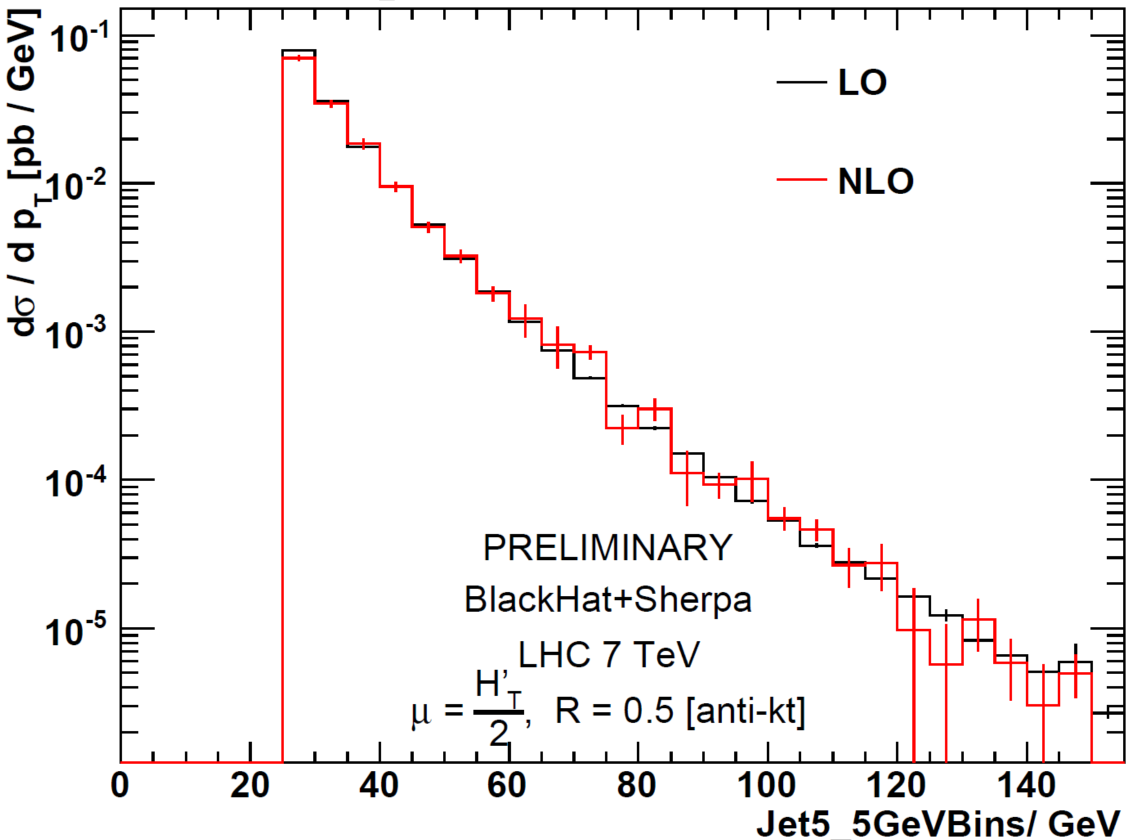
Same construction works in gravity

Brandhuber, Travaglini, Spence; Cachazo, Svrcek;
Benincasa, Boucher-Veronneau, Cachazo; Arkani-Hamed and Kaplan, Hall

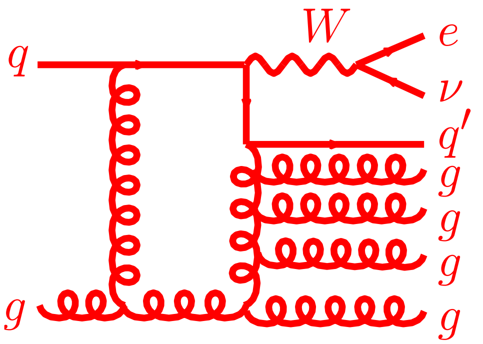
Preliminary W + 5 Jets in NLO QCD

ZB, Dixon, Febres Cordero, Hoeche, Ita, Kosower, Maitre, Ozeren [BlackHat collaboration]

P_T spectrum of 5th jet



Uses leading-color approximation. Good to 3% for W + 1,2,3,4 jets



W + 3,4 jets done only via unitarity method.

- **A new level for "state of the art". First NLO QCD 2→6 process!**
- **People at ATLAS promise to immediately compare to data when complete. Particularly important background to $\bar{t}\,t$ production.**

The Structure of Multiloop Gauge and Gravity Amplitudes

Gravity vs Gauge Theory

Consider the gravity Lagrangian

flat metric

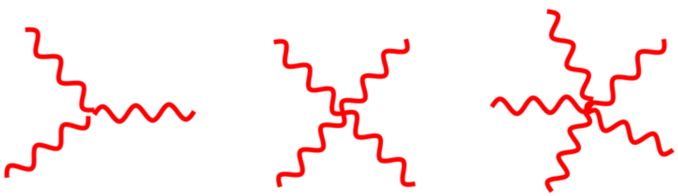
graviton field

$$L_{\text{gravity}} = \frac{2}{\kappa^2}\sqrt{-g}\,R \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

metric

$$\kappa^2 = 32\pi G_{\text{Newton}}$$

Infinite number of complicated interactions



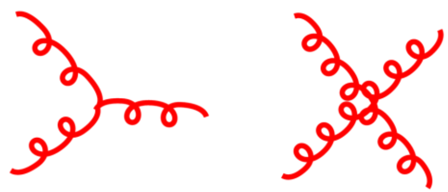
$+ \cdots$

terrible mess

Non-renormalizable

Compare to Yang-Mills Lagrangian on which QCD is based

$$L_{\text{YM}} = \frac{1}{g^2}F^2$$

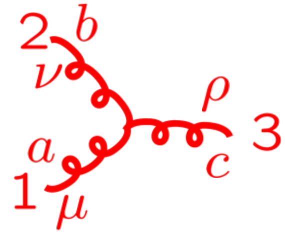


Only three and four point interactions

Gravity seems so much more complicated than gauge theory.

Three Vertices

Standard Feynman diagram approach.



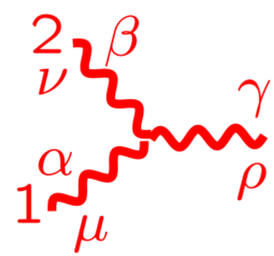
Three-gluon vertex:

$$V_{3\,\mu\nu\sigma}^{abc} = -gf^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho+\eta_{\nu\rho}(k_1-k_2)_\mu+\eta_{\rho\mu}(k_1-k_2)_\nu)$$

$$k_i^2 = E_i^2 - \vec{k}_i^2 \neq 0$$

Three-graviton vertex:

$$\begin{aligned} &G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) = \\ &\quad \mathrm{sym}[-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) \\ &\qquad + P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) \\ &\qquad + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) \\ &\qquad + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})] \end{aligned}$$



About 100 terms in three vertex

Naïve conclusion: Gravity is a nasty mess.

Definitely not a good approach.

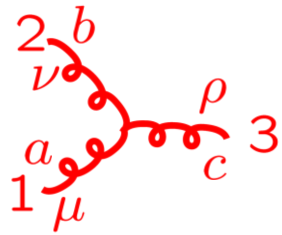
Simplicity of Gravity Amplitudes

People were looking at gravity the wrong way. On-shell viewpoint much more powerful.

***On-shell* three vertices contains all information:**

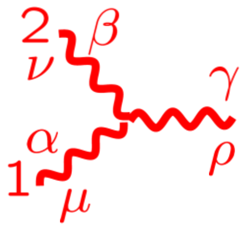
$$k_i^2 = 0$$

gauge theory:



$$-gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$

gravity:



$$i\kappa(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic}) \times (\eta_{\alpha\beta}(k_1 - k_2)_\gamma + \text{cyclic})$$

double copy of Yang-Mills vertex.

- **Using modern on-shell methods, any gravity scattering amplitude constructible solely from *on-shell* 3 vertex.**
- **Higher-point vertices irrelevant! BCFW recursion for trees, BDDK unitarity method for loops.**

Gravity vs Gauge Theory

Consider the gravity Lagrangian

$$L_{\text{gravity}} = \frac{2}{\kappa^2}\sqrt{-g}\,R \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

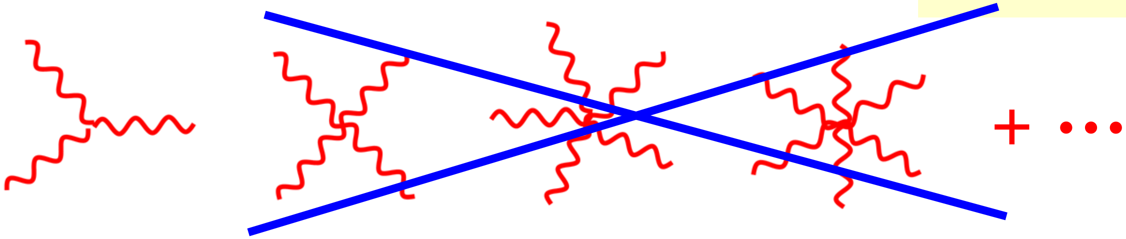
metric

flat metric

graviton field

$$\kappa^2 = 32\pi\, G_{\text{Newton}}$$

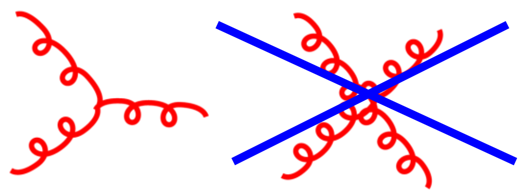
Infinite number of irrelevant interactions!



Simple relation to gauge theory

Compare to Yang-Mills Lagrangian

$$L_{\text{YM}} = \frac{1}{g^2} F^2$$



Only three-point interactions

Gravity seems ~~so much~~ no more complicated than gauge theory.

Duality Between Color and Kinematics

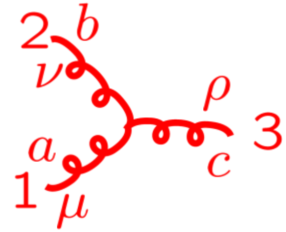
ZB, Carrasco, Johansson (BCJ)

coupling constant

color factor

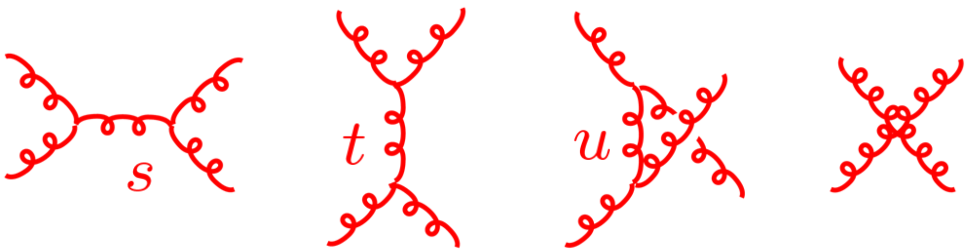
momentum dependent kinematic factor

$$-gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$



Color factors based on a Lie algebra: $[T^a, T^b] = i f^{abc} T^c$

Jacobi Identity $f^{a_1 a_2 b} f^{b a_4 a_3} + f^{a_4 a_2 b} f^{b a_3 a_1} + f^{a_4 a_1 b} f^{b a_2 a_3} = 0$



Use 1 = *s/s* = *t/t* = *u/u* to assign 4-point diagram to others.

$$\mathcal{A}_4^{\text{tree}} = g^2 \left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u} \right)$$

$s = (k_1 + k_2)^2$ $\quad u = (k_1 + k_3)^2$

$t = (k_1 + k_4)^2$

Color factors satisfy Jacobi identity: $c_u = c_s - c_t$

Numerator factors satisfy similar identity: $n_u = n_s - n_t$

Color and kinematics satisfy the same identity

Duality Between Color and Kinematics

BCJ

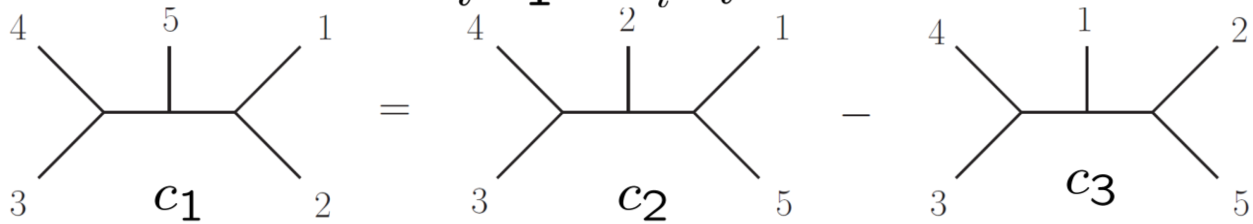
Consider five-point tree amplitude:

color factor

kinematic numerator factor

$$\mathcal{A}_5^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i\, n_i}{\prod_{\alpha_i} p_i^2}$$

Feynman propagators



$$c_1 \equiv f^{a_3 a_4 b} f^{b a_5 c} f^{c a_1 a_2}, \quad c_2 \equiv f^{a_3 a_4 b} f^{b a_2 c} f^{c a_1 a_5}, \quad c_3 \equiv f^{a_3 a_4 b} f^{b a_1 c} f^{c a_2 a_5}$$

$$n_i \sim k_4 \cdot k_5\, k_2 \cdot \varepsilon_1\, \varepsilon_2 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5 + \cdots$$

$$\boxed{c_1 - c_2 + c_3 = 0 \Leftrightarrow n_1 - n_2 + n_3 = 0}$$

Claim: We can always find a rearrangement so color and kinematics satisfy the *same* Jacobi constraint equations.

Nontrivial constraints on amplitudes in field theory and string theory

BCJ, Bjerrum-Bohr, Feng,Damgaard, Vanhove, ; Mafra, Stieberger, Schlotterer; Tye and Zhang; Feng, Huang, Jia; Chen, Du, Feng; Du ,Feng, Fu; Naculich, Nastase, Schnitzer

BCJ

Gravity and Gauge Theory

kinematic numerator

color factor

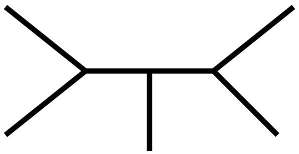
gauge theory: $$\frac{1}{g^{n-2}}\mathcal{A}_n^{\text{tree}}(1,2,3,\ldots,n) = \sum_i \frac{n_i\, c_i}{\prod_{\alpha_i} p^2_{\alpha_i}}$$

sum over diagrams with only 3 vertices

$$c_i \sim f^{a_1 a_2 b_1} f^{b_1 b_2 a_5} f^{b_2 a_4 a_5}$$

Assume we have:

$$c_1 + c_2 + c_3 = 0 \quad \Leftrightarrow \quad n_1 + n_2 + n_3 = 0$$



Then: $c_i \Rightarrow \tilde{n}_i$ **kinematic numerator of second gauge theory**

Proof: ZB, Dennen, Huang, Kiermaier

gravity: $$\boxed{-i\left(\frac{2}{\kappa}\right)^{(n-2)} \mathcal{M}_n^{\text{tree}}(1,2,\ldots,n) = \sum_i \frac{n_i\, \tilde{n}_i}{\prod_{\alpha_i} p^2_{\alpha_i}}}$$

Gravity numerators are a double copy of gauge-theory ones!

This works for ordinary Einstein gravity and susy versions!

Cries out for a unified description of the sort given by string theory!

Gravity From Gauge Theory

$$-i\left(\frac{2}{\kappa}\right)^{(n-2)} \mathcal{M}_n^{\rm tree}(1,2,\ldots,n) = \sum_i \frac{n_i\,\tilde{n}_i}{\prod_{\alpha_i} p^2_{\alpha_i}}$$

n $\tilde{n}$

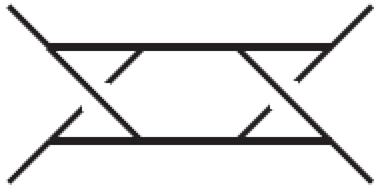
N = 8 sugra: (N = 4 sYM)× (N = 4 sYM)
N = 6 sugra: (N = 4 sYM)× (N = 2 sYM)
N = 4 sugra: (N = 4 sYM)× (N = 0 sYM)
N = 0 sugra: (N = 0 sYM)× (N = 0 sYM)

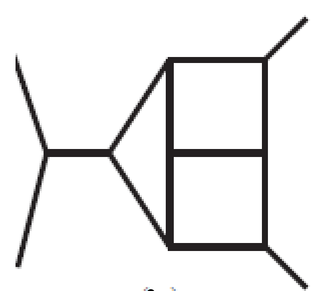
N = 0 sugra: graviton + antisym tensor + dilaton

Claim: Once the gauge theory integrands is arranged to satisfy duality obtaining the corresponding gravity integrands at *any* loop order is trivial!

Loop-Level Generalization

ZB, Carrasco, Johansson (2010)





$$c_i + c_j + c_k = 0$$
$$n_i + n_j + n_k = 0$$

sum is over diagrams

kinematic numerator

color factor

$$\frac{(-i)^L}{g^{n-2+2L}}\mathcal{A}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

gauge theory

propagators

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}}\mathcal{M}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

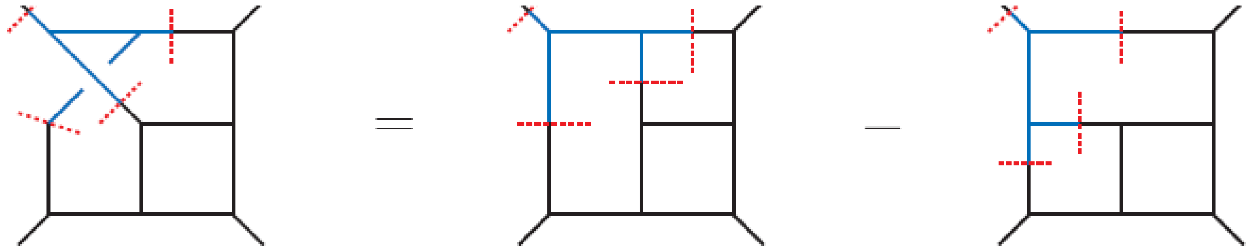
gravity

symmetry factor

Loop-level conjecture is identical to tree-level one except for symmetry factors and loop integration.
Double copy works if numerator satisfies duality.

Nonplanar from Planar

BCJ



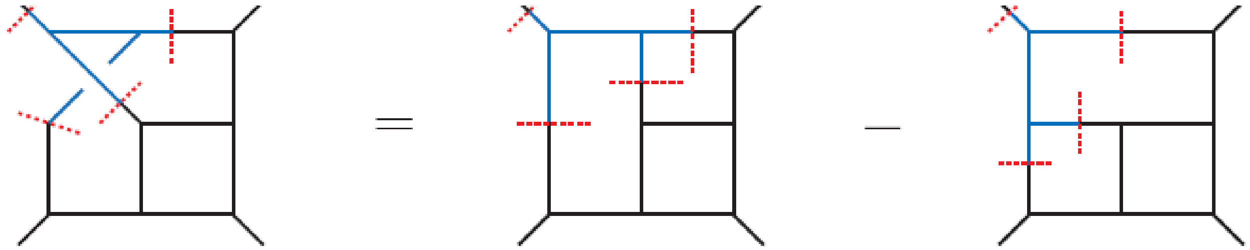
$$c_k = c_i - c_j$$
$$n_k = n_i - n_j$$

Planar determines nonplanar

- **We can carry advances from planar sector to the nonplanar sector.**
- **Only at level of the integrands, so far, but bodes well for the future.**

Gravity integrands are free!

BCJ



$$c_k = c_i - c_j$$
$$n_k = n_i - n_j$$

If you have a set of duality satisfying numerators. To get:

gauge theory ⟶ gravity theory

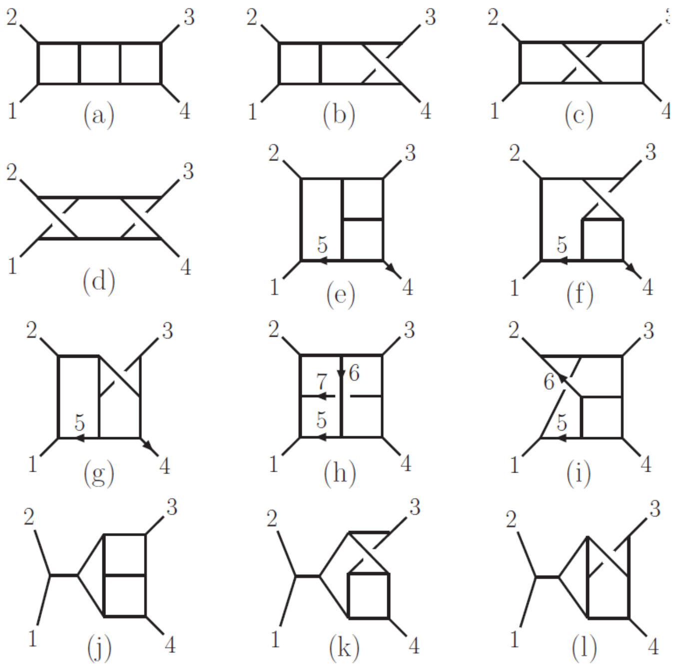
simply take

color factor ⟶ kinematic numerator

Gravity loop integrands are free!

Explicit Three-Loop Check

ZB, Carrasco, Johansson (2010)



$$c_i = c_j - c_k \Rightarrow n_i = n_j - n_k$$

For $\mathcal{N}$=4 sYM we have the abililty to go to high loop orders. Go to 3 loops. (1 & 2 loops works.)

Calculation very similar to earlier one with Dixon and Roiban, except now make the duality manifestly

Integral $I^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8 \text{ supergravity}}$) numerator
(a)–(d)	s^2
(e)–(g)	$\left(s\,(-\tau_{35}+\tau_{45}+t) - t\,(\tau_{25}+\tau_{45}) + u\,(\tau_{25}+\tau_{35}) - s^2 \right)/3$
(h)	$\left(s\,(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u) + t\,(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17}) + s^2 \right)/3$
(i)	$\left(s\,(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t) + t\,(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46}) + u\,\tau_{25} + s^2 \right)/3$
(j)-(l)	$s(t-u)/3$

$$\tau_{ij} = 2k_i \cdot l_j$$

- **Duality works!**
- **Double copy works!**

One diagram to rule them all

ZB, Carrasco, Johansson (2010)

***N* = 4 super-Yang-Mills integrand**

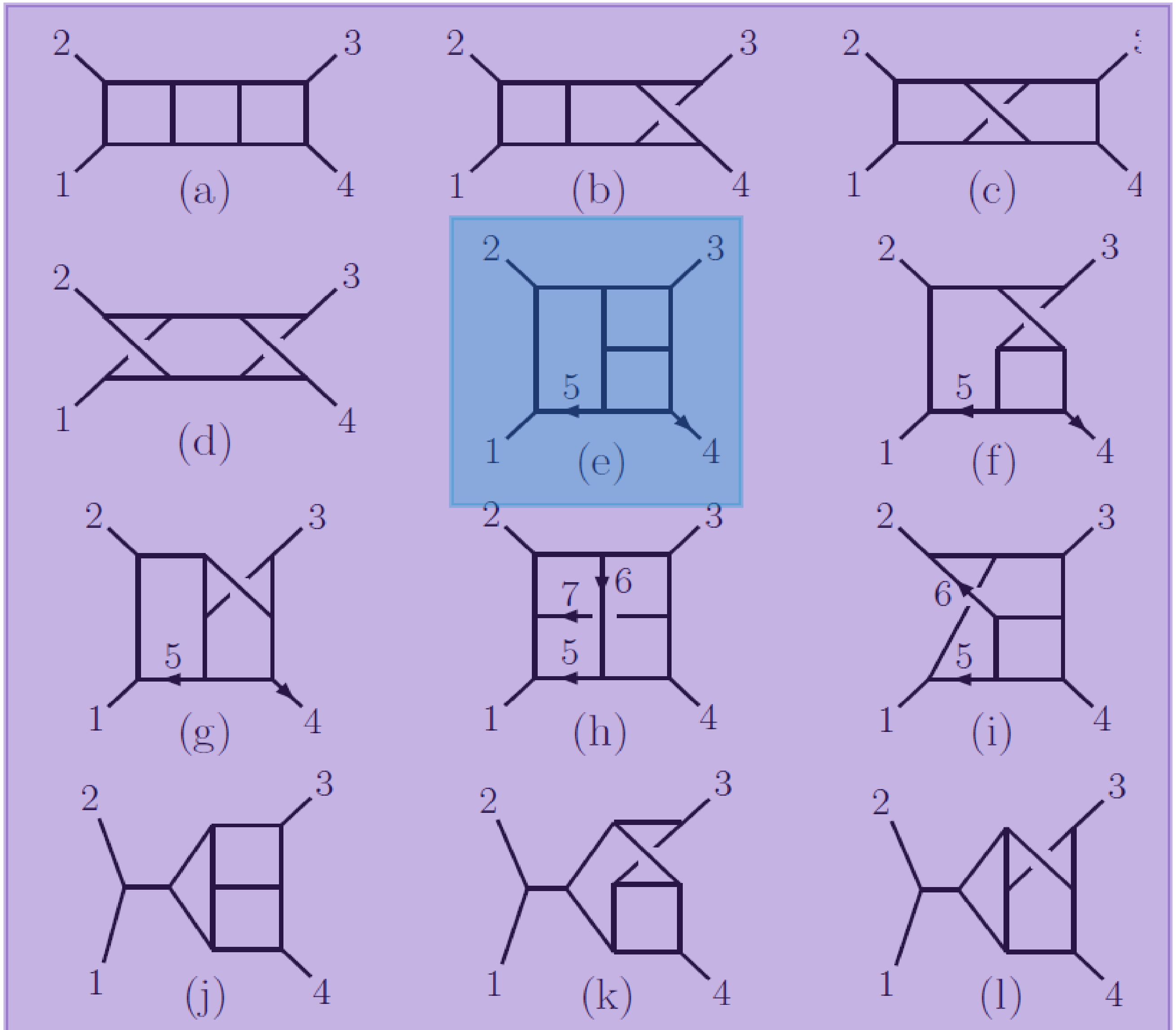


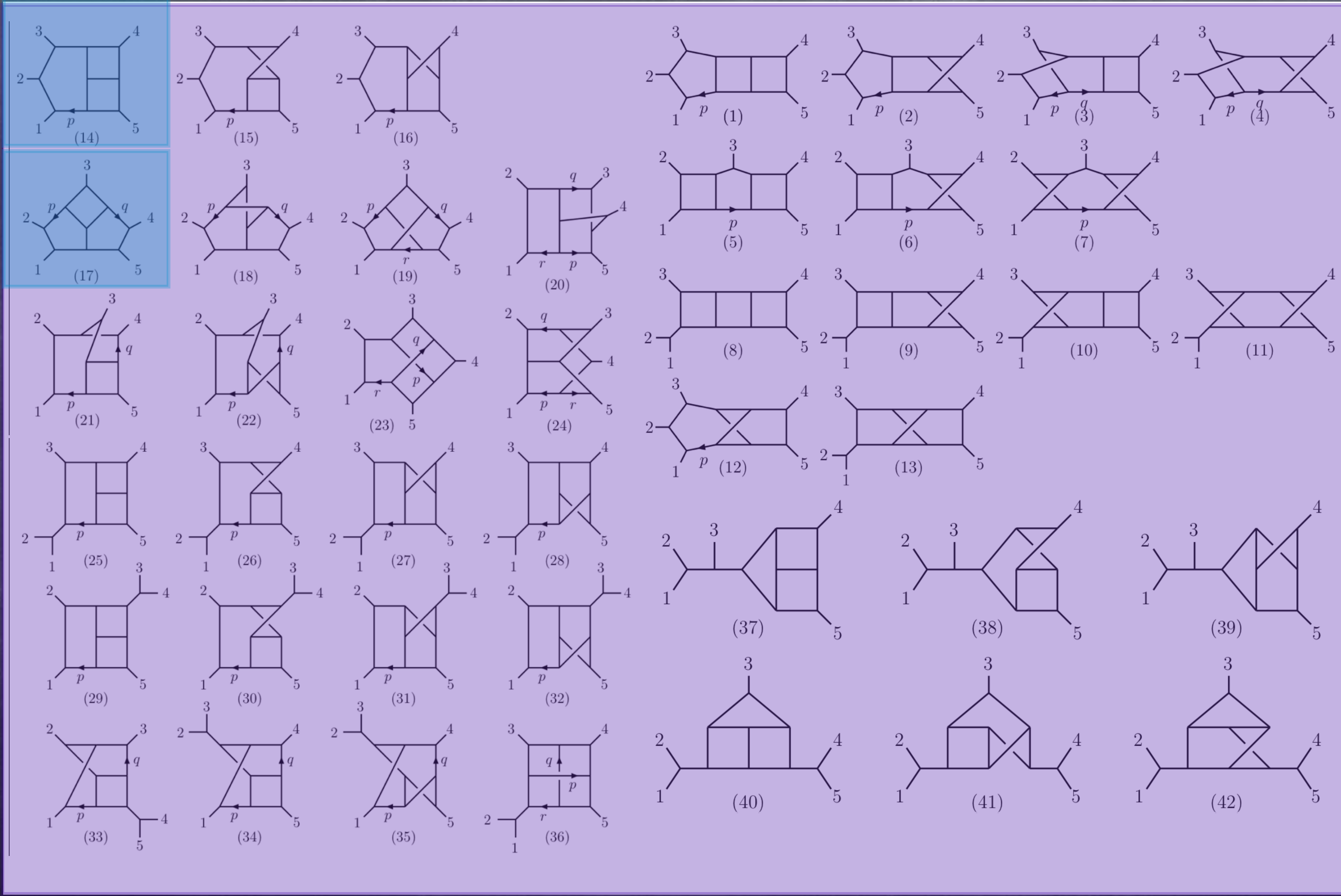
Diagram (e) is the master diagram.

Determine the master integrand in proper form and duality gives all others.

***N* = 8 sugra given by double copy.**

Five point 3-loop N=4 SYM & N=8 SUGRA

Carrasco, Johansson



Generalized Gauge Invariance

BCJ
ZB, Dennen, Huang, Kiermaier
Tye and Zhang

gauge theory

$$\frac{(-i)^L}{g^{m-2+2L}}\mathcal{A}_m^{\text{loop}} = \sum_j \int \frac{d^{DL}p}{(2\pi)^{DL}} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

$$n_i \to n_i + \Delta_i \qquad \sum_j \int \frac{d^{DL}p}{(2\pi)^{DL}} \frac{1}{S_j} \frac{\Delta_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2} = 0$$

$$(c_\alpha + c_\beta + c_\gamma) f(p_i) = 0$$

Above is just a definition of generalized gauge invariance

gravity

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}}\mathcal{M}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

$$n_i \to n_i + \Delta_i \qquad \sum_j \int \frac{d^{DL}p}{(2\pi)^{DL}} \frac{1}{S_j} \frac{\Delta_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2} = 0$$

- **Gravity inherits generalized gauge invariance from gauge theory!**
- **Double copy works even if only one of the two copies has duality manifest!**
- **Used to find expressions for $N \geq 4$ supergravity amplitudes at 1, 2 loops.** ZB, Boucher-Veronneau and Johansson; Boucher-Veroneau and Dixon

Application: UV Properties of Gravity

UV properties of gravity

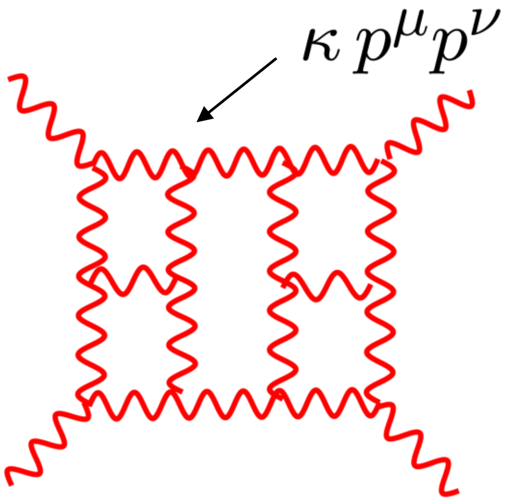
Study of UV divergences has a long history starting with ‘t Hooft and Veltman. I will only skim over the history here.

We study this issue not because we think a finite theory of gravity immediately gives a good description of Nature—long list of nontrivial issues.

We study it because the theory can only be finite due to *a fantastic* new symmetry or dynamical mechanism.

Power Counting at High Loop Orders

$\kappa = \sqrt{32\pi G_N}$ ⟵ Dimensionful coupling



Gravity: $$\int \prod_{i=1}^{L} \frac{dp_i^D}{(2\pi)^D} \frac{(\kappa\, p_j^\mu p_j^\nu)\cdots}{\text{propagators}}$$

Gauge theory: $$\int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{(g\, p_j^\nu)\cdots}{\text{propagators}}$$

Extra powers of loop momenta in numerator means integrals are badly behaved in the UV.

Non-renormalizable by power counting.

Reasons to focus on *N* = 8 supegravity:

- **With more susy expect better UV properties.**
- **High symmetry implies technical simplicity.**

Opinions from the 80's

Unfortunately, in the absence of further mechanisms for cancellation, the analogous $N = 8$ $D = 4$ supergravity theory would seem set to diverge at the three-loop order.

Howe, Stelle (1984)

It is therefore very likely that all supergravity theories will diverge at three loops in four dimensions. … The final word on these issues may have to await further explicit calculations.

Marcus, Sagnotti (1985)

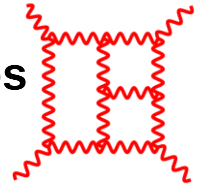
The idea that *all* supergravity theories diverge (at three loops) has been widely accepted for over 25 years

R^4 **is expected counterterm**

Feynman Diagrams for Gravity

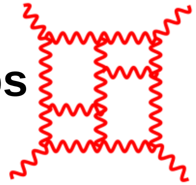
Suppose we want to check if opinions are true using Feynman digrams

3 loops



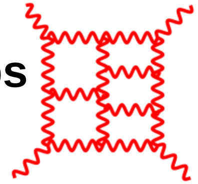
$\sim 10^{20}$ TERMS

4 loops



$\sim 10^{26}$ TERMS

5 loops



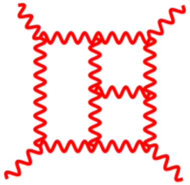
$\sim 10^{31}$ TERMS

Has as never been calculated via Feynman diagrams.

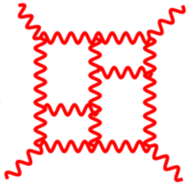
- **Calculations to settle this seemed utterly hopeless!**
- **Seemed destined for dustbin of undecidable questions.**

Unitarity Method + Double Copy

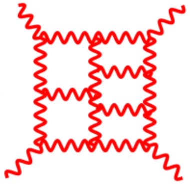
For $N = 8$ supergravity.

3 loops  ~10 TERMS

Calculation doable on a blackboard

4 loops  $\sim 10^2$ TERMS

Much more manageable!

5 loops  $\sim 10^3$ TERMS

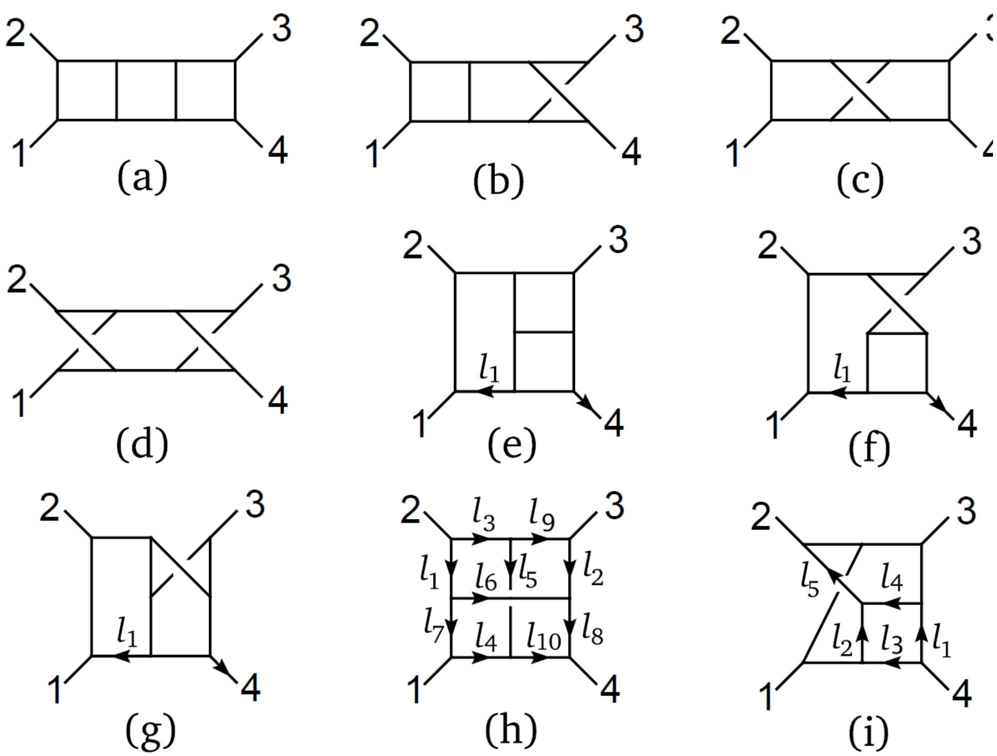
(Not yet completed)

Complete Three-Loop Result

ZB, Carrasco, Dixon, Johansson, Kosower, Roiban; hep-th/0702112
ZB, Carrasco, Dixon, Johansson, Roiban arXiv:0808.4112 [hep-th]

Obtained via on-shell unitarity method:

$$\tau_{ij} = 2k_i \cdot k_j$$



Three loops is not only ultraviolet finite it is "superfinite"— finite for *D* < 6.

All UV cancellations exposed manifestly

It's very finite!

A More Recent Opinion

Back in 2009 Bossard, Howe and Stelle had a look at the question of how much supersymmetry can tame UV divergences.

In particular … suggest that maximal supergravity is likely to diverge at four loops in *D* = 5 and at five loops in *D* = 4 …

Bossard, Howe, Stelle (2009)

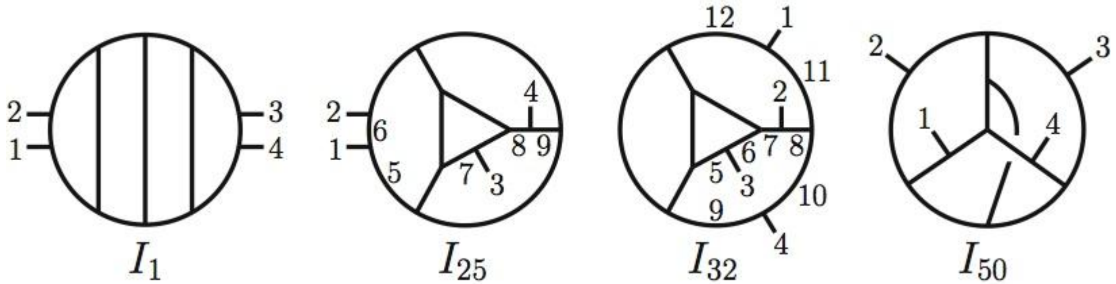
Bottles of wine were at stake!

We had the tools to collect the wine!

Four-Loop Amplitude Construction

ZB, Carrasco, Dixon, Johansson, Roiban

Get 50 distinct diagrams or integrals (ones with two- or three-point subdiagrams not needed).



$$M_4^{4\text{-loop}} = \left(\frac{\kappa}{2}\right)^{10} stu M_4^{\text{tree}} \sum_{S_4} \sum_{i=1}^{50} c_i I_i$$

Integral

symmetry factor

leg perms

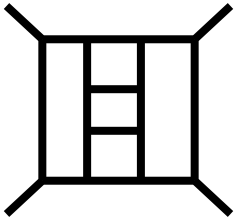
UV finite for $D < 5.5$
It's very finite!

Originally took more than a year.
Today with the double copy we can reproduce it in a couple of days! Another non-trivial example.

Status of counterterms a month ago

Various papers argued that in N = 8 supergravity trouble starts at 5 loops and by 7 loops we have valid UV counterterm in D = 4 under all known symmetries (suggesting divergences) .

Bossard, Howe, Stelle; Elvang, Freedman, Kiermaier; Green, Russo, Vanhove ; Green and Bjornsson ; Bossard , Hillmann and Nicolai; Ramond and Kallosh; Broedel and Dixon; Elvang and Kiermaier; Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger



For N = 8 sugra in D = 4:

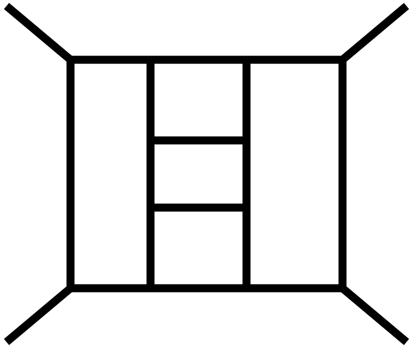
- **All counterterms ruled out until 7 loops!**
- **Candidate 7 loop superspace volume counterterm vanishes.**
- **But D^8R^4 apparently available at 7 loops (1/8 BPS) under all known symmetries.** Bossard, Howe, Stelle and Vanhove

For N = 4 sugra in D = 4:

- **R^4 apparently available at 3 loops (1/4 BPS) under all known symmetries. (No known nonrenormalization theorems).**
- **A three-loop diverges expected by everyone…**

Status of 5 Loop Calculation

ZB, Carrasco, Johannson, Roiban



450 such diagrams with ~100s terms each

We have corresponding N = 4 sYM calculation complete…

Stay tuned! We are going to find out!

Place your bets:

At 5 loops in D=24/5 does N = 8 supergravity diverge?

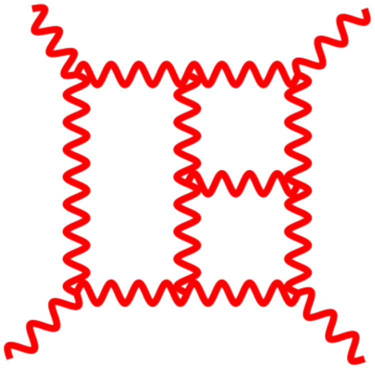
It's very likely game over in D = 4 if we find a divergence here.

Kelly Stelle:
British wine
"It will diverge"

Zvi Bern:
California wine
"It won't diverge"

$N = 4$ supergravity

$N = 4$ sugra at 3 loops ideal test case.

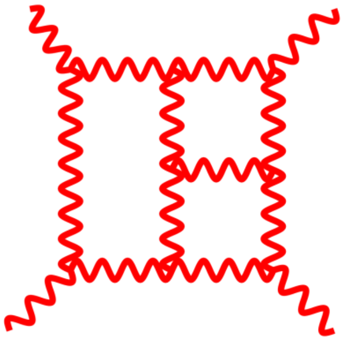


Consensus has it that a valid R^4 counterterm existed.

Bossard, Howe, Stelle; Bossard, Howe, Stelle, Vanhove

- **Need to maximize the susy for simplicity.**
- **Need to minimize the susy to lower the loop order where we might find potential divergences.**
- **BCJ duality gives us the power to do the calculation.**

N = 4 supergravity



A no lose calculation:
Either we find first example of a divergence or once again we show an expected divergence is not present!

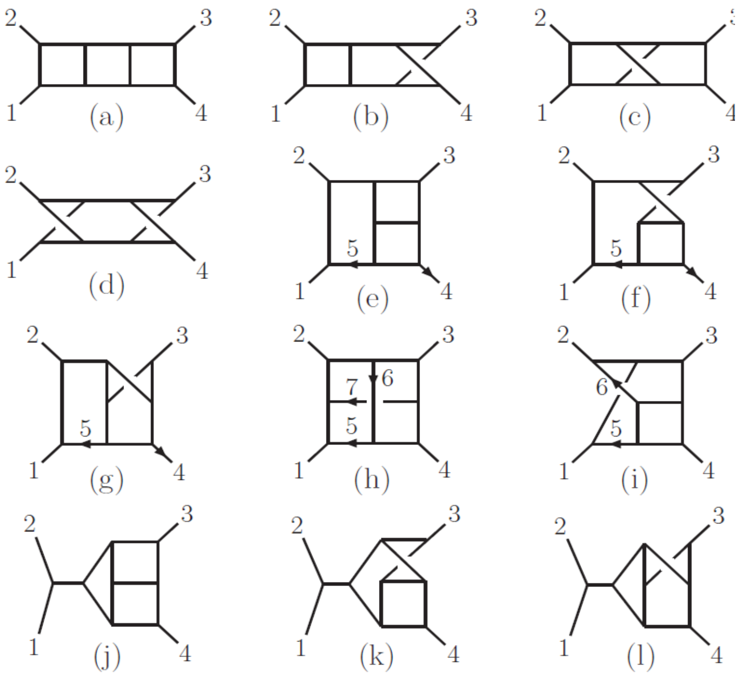
One year everyone believed that supergravity was finite. The next year the fashion changed and everyone said that supergravity was bound to have divergences even though none had actually been found. — ***Stephen Hawking, 1994***

To this day no one has ever proven that *any* pure supergravity diverges in D = 4.

Three-loop construction

ZB, Davies, Dennen, Huang

$N = 4$ sugra : ($N = 4$ sYM) x ($N = 0$ YM)



- **For $N = 4$ sYM copy use known BCJ representation.**
- **What representation should we use for pure YM side?**

Integral $I^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8}$ supergravity) numerator
(a)–(d)	s^2
(e)–(g)	$\left(s\,(-\tau_{35}+\tau_{45}+t) - t\,(\tau_{25}+\tau_{45}) + u\,(\tau_{25}+\tau_{35}) - s^2 \right)/3$
(h)	$\left(s\,(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u) \right.$ $\left. +t\,(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17}) + s^2 \right)/3$
(i)	$\left(s\,(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t) \right.$ $\left. +t\,(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46}) + u\,\tau_{25} + s^2 \right)/3$
(j)-(l)	$s(t-u)/3$

$N = 4$ sYM integrand

Three-loop $N = 4$ supergravity

What is a convenient representation for pure YM copy?

Answer: Feynman diagrams.

This case is very special

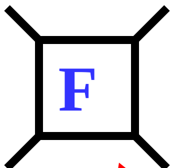
- **We can drop all Feynman diagrams where corresponding $N = 4$ numerators vanish.**
- **We need only the leading UV parts.**
- **Completely straightforward. Faster to just do it than to argue about which method might be better.**

Multiloop $N = 4$ supergravity

Does it work? Test at 1, 2 loops

All supergravities finite at 1,2 loops

One-loop: keep only box Feynman diagrams

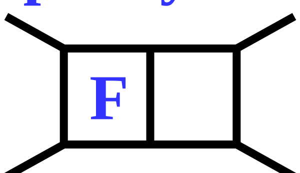
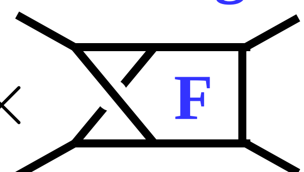
$st A_4^{\text{tree}} \times$  + perms $= \frac{0}{\epsilon} =$ finite

***N* = 4 sYM box numerator**

***N* = 0 Feynman diagram, including ghosts**

Becomes gauge invariant after permutation sum.

Two-loop: keep only double box Feynman diagrams

$s^2 t A_4^{\text{tree}} \times$  $+\, s^2 t A_4^{\text{tree}} \times$  + perms $= \frac{0}{\epsilon}$

Feynman diagram including ghosts

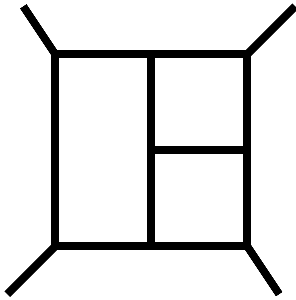
Get correct results. Who would have imagined gravity is this simple?

Three-Loop Construction

Now apply the construction to three loops.

***N* = 4 sugra : (*N* = 4 sYM) x (*N* = 0 YM)**

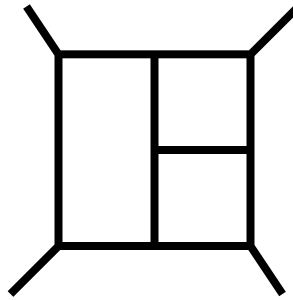
***N* = 4 sYM**



$\sim l \cdot k\, s^2 t A_4^{\rm tree}$

BCJ representation

pure YM



$\sim (\varepsilon_i \cdot l)^4\ l^4$

Feynman representation

***N* = 4 sugra linear divergent**

$$\int (d^D l)^3 \frac{k^7 l^9}{l^{20}}$$

simple to see finite for *N*=5,6 sugra

Pure YM 4 point amplitude has never been done at three loops.

Numerator: $k^7 l^9 + k^8\, l^8$ **+ finite**

log divergent

Will series expand in external momenta *k*

Dealing With Subdivergences

Marcus, Sagnotti (1984)

The problem was solve nearly 30 years ago.

Recursively subtract all subdivergences.

regulator dependent

reparametrize subintegral

$$\mathcal{S}\left[\int \prod_{i=1}^{L} dp_i\, I\right] \equiv \text{Div}\left[\int \prod_{i=1}^{L} dp_i\, I\right] - \sum_{l=1}^{L-1} \sum_{\substack{l-\text{loop}\\ \text{subloops}}} \text{Div}\left[\int \prod_{j=l+1}^{L} dp'_j\, \mathcal{S}\left[\int \prod_{i=1}^{l} dp'_i\, I\right]\right]$$

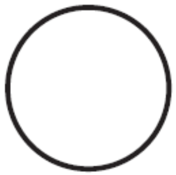
Regulator Independent

Nice consistency check: all log(*m*) terms must cancel

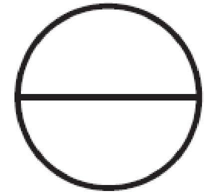
Extracting UV divergence in the presence of UV subdivergences and IR divergences is a well understood problem.

Integral Basis

Using FIRE we obtain a basis of integrals:

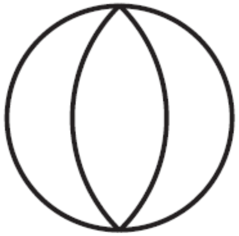


$$(m^2)^{1-\epsilon}\left(\frac{1}{\epsilon}+1+\left(1+\frac{1}{2}\zeta_2\right)\epsilon\right)$$



$$(m^2)^{1-2\epsilon}\left(\frac{3}{2\epsilon^2}+\frac{9}{2\epsilon}+\frac{21}{2}+\frac{3}{2}\zeta_2-2c\right)$$

$$(m^2)^{1-3\epsilon}\left(\frac{1}{\epsilon^3}+\frac{17}{3\epsilon^2}+\left(\frac{67}{3}+\frac{3}{2}\zeta_2-4c\right)\frac{1}{\epsilon}\right)$$



$$(m^2)^{2-3\epsilon}\left(\frac{2}{\epsilon^3}+\frac{23}{3\epsilon^2}+\left(\frac{35}{2}+3\zeta_2\right)\frac{1}{\epsilon}\right)$$

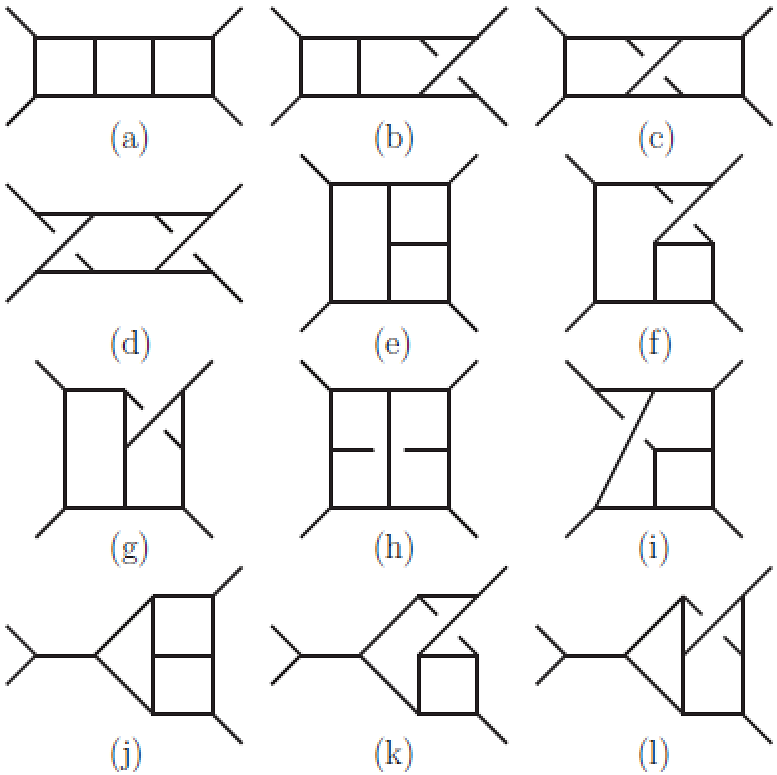
$$(m^2)^{-3\epsilon}\left(\frac{2\zeta_3}{\epsilon}\right)$$

$$c=\sqrt{3}\,\mathrm{Im}\left(\mathrm{Li}_2(e^{i\pi/3})\right)$$

Use Mellin-Barnes resummation of residues method of Davydychev and Kalmykov on all but last integral. Last one doable by staring at paper from Grozin or Smirnov's book (easy because no subdivergences).

The $N = 4$ Supergravity UV Cancellation

ZB, Davies, Dennen, Huang



Graph	(divergence)$/(\langle 12\rangle^2 [34]^2 st A^{\text{tree}} (\frac{\kappa}{2})^8)$
(a)-(d)	0
(e)	$\frac{263}{768}\frac{1}{\epsilon^3} + \frac{205}{27648}\frac{1}{\epsilon^2} + \left(-\frac{5551}{768}\zeta_3 + \frac{326317}{110592}\right)\frac{1}{\epsilon}$
(f)	$-\frac{175}{2304}\frac{1}{\epsilon^3} - \frac{1}{4}\frac{1}{\epsilon^2} + \left(\frac{593}{288}\zeta_3 - \frac{217571}{165888}\right)\frac{1}{\epsilon}$
(g)	$-\frac{11}{36}\frac{1}{\epsilon^3} + \frac{2057}{6912}\frac{1}{\epsilon^2} + \left(\frac{10769}{2304}\zeta_3 - \frac{226201}{165888}\right)\frac{1}{\epsilon}$
(h)	$-\frac{3}{32}\frac{1}{\epsilon^3} - \frac{41}{1536}\frac{1}{\epsilon^2} + \left(\frac{3227}{2304}\zeta_3 - \frac{3329}{18432}\right)\frac{1}{\epsilon}$
(i)	$\frac{17}{128}\frac{1}{\epsilon^3} - \frac{29}{1024}\frac{1}{\epsilon^2} + \left(-\frac{2087}{2304}\zeta_3 - \frac{10495}{110592}\right)\frac{1}{\epsilon}$
(j)	$-\frac{15}{32}\frac{1}{\epsilon^3} + \frac{9}{64}\frac{1}{\epsilon^2} + \left(\frac{101}{12}\zeta_3 - \frac{3227}{1152}\right)\frac{1}{\epsilon}$
(k)	$\frac{5}{64}\frac{1}{\epsilon^3} + \frac{89}{1152}\frac{1}{\epsilon^2} + \left(-\frac{377}{144}\zeta_3 + \frac{287}{432}\right)\frac{1}{\epsilon}$
(l)	$\frac{25}{64}\frac{1}{\epsilon^3} - \frac{251}{1152}\frac{1}{\epsilon^2} + \left(-\frac{835}{144}\zeta_3 + \frac{7385}{3456}\right)\frac{1}{\epsilon}$

Sum over diagrams is gauge invariant

All divergences cancel completely!

Surprise: it's actually UV finite

Explanations?

Key Question: Is there an ordinary symmetry explanation for this? Or is something extraordinary happening?

Quantum corrected duality current nonconservation. Kallosh (2012)

Non-renormalization understanding from heterotic string. Tourkine and Vanhove (2012)

- **Is there a simple way to understand our result?**
- **What role does the global U(1) anomaly play?** Marcus (1985)
- **Why don't similar cancellations happen for *N* = 8 supergravity, killing potential 7 loop counterterm?**

Answering questions

- **Is the cancellation special to $N = 4$ supergravity?**
  - **We need to know result of $N = 8$ supergravity 4 pt 5 loop calculation. This is a critical calculation.**
- **Four loops: D^2R^4 counterterm available.**
  - **Full superspace integral – analogous to 8-loop counterterm in $N = 8$ sugra.**
  - **Doable because $N = 4$ sYM BCJ representation exists.**

- **Three-loop five points: ϕR^4 counterterm might be available.**
  - **Doable because BCJ representation exists.**

(Carrasco & Johansson, unpublished)

A critical problem for gravity is to find BCJ representations for as many gauge-theory amplitudes as possible…

Summary

- **Unitarity method is by now a standard tool for state-of-the-art amplitude calculations.**
- **A new duality conjectured between color and kinematics. Locks down the multiloop structure of integrands of amplitudes.**
- **When duality between color and kinematics manifest, gravity integrands follow immediately from gauge-theory ones.**
- **Double copy gives us a powerful way to explore the UV properties of gravity theories.**
- **$\mathcal{N} = 4$ sugra has no three-loop four-point divergence, contrary to expectations from symmetry considerations. A surprise!**
- **Power counting using known symmetries and their known consequences can be misleading.**

$\mathcal{N} = 8$ supergravity may turn out to be the first example of a $D = 4$ unitary point-like perturbatively UV finite theory of gravity. Demonstrating this remains a challenge.

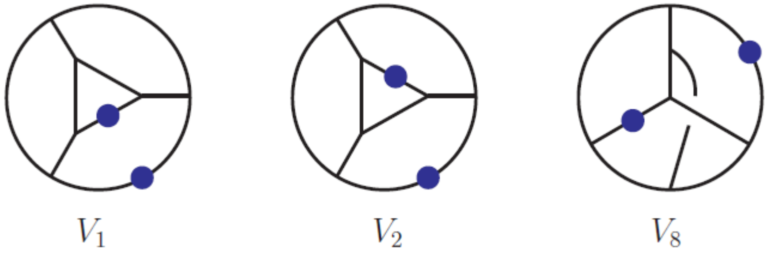
Extra Slides

New Four Loop Surprise

ZB, Carrasco, Dixon, Johansson, Roiban

doubled propagator

Critical dimension *D* =11/2.
Express UV divergences
in terms of vacuum like integrals.



gauge theory

$$\mathcal{A}_4^{(4)}(1,2,3,4)\Big|_{\rm pole}^{SU(N_c)} = -6\, g^{10}\, \mathcal{K}\, N_c^2 \Big(N_c^2\, V_1 + 12\, \boxed{(V_1 + 2\,V_2 + V_8)}\Big)$$
$$\times \Big(s\,(\,\mathrm{Tr}_{1324} + \mathrm{Tr}_{1423}) + t\,(\,\mathrm{Tr}_{1243} + \mathrm{Tr}_{1342}) + u\,(\,\mathrm{Tr}_{1234} + \mathrm{Tr}_{1432})\Big)$$

same divergence!!

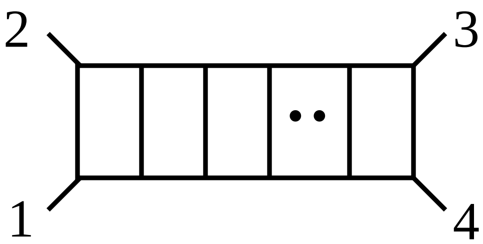
gravity

$$\mathcal{M}_4^{(4)}\Big|_{\rm pole} = -\frac{23}{8}\left(\frac{\kappa}{2}\right)^{10} stu(s^2+t^2+u^2)^2\, M_4^{\rm tree}\, \boxed{(V_1 + 2V_2 + V_8)}$$

- **Gravity UV divergence is directly proportional to subleading color single-trace divergence of *N* = 4 super-Yang-Mills theory!**
- **Same happens at 1-3 loops.**

$\mathcal{N}=8$ L-Loop UV Cancellations

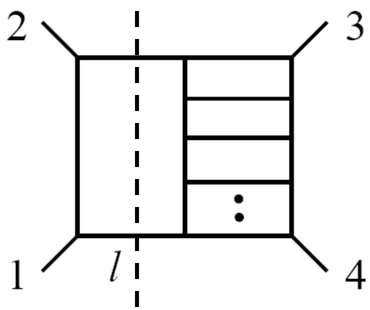
ZB, Dixon, Roiban



$$[(k_1+k_2)^2]^{2(L-2)}$$

numerator factor

From 2 particle cut:

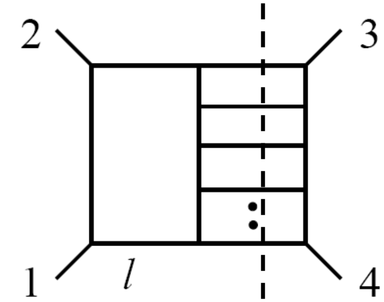


1 in $\mathcal{N}=4$ YM

$$[(l+k_4)^2]^{2(L-2)}$$

numerator factor

L-particle cut



- **Numerator violates one-loop "no-triangle" property.**
- **Too many powers of loop momentum in one-loop subamplitude.**
- **After cancellations behavior is same as in $\mathcal{N}=4$ Yang-Mills!**

- **UV cancellation exist to *all* loop orders! (not a proof of finiteness)**
- **These *all-loop* cancellations *not* explained by any known supersymmetry arguments.**
- **Existence of these cancellations motivates our calculations.**

Summary of Tree Checks and Understanding

1) Nontrivial consequences for tree amplitudes BCJ

$$A_5^{\text{tree}}(1,3,4,2,5) = \frac{-s_{12}s_{45}A_5^{\text{tree}}(1,2,3,4,5) + s_{14}(s_{24}+s_{25})A_5^{\text{tree}}(1,4,3,2,5)}{s_{13}s_{24}}$$

Proven using on-shell recursion and also string theory.

Bjerrum-Bohr, Damgaard,Vanhove; Steiberger; Sondergaard,; Chen, Du, Feng

2) Proof of gravity double-copy tree formula.

ZB, Dennen, Huang, Kiermaier

3) Reasonable string-theory understanding of duality.

Tye and Zhang; Mafra, Schlotterer Stieberger Bjerrum-Bohr, Damgaard, Vanhove, Sondergaard.

4) Explicit formulas for numerators in terms of amplitudes.

Kiermaier; Bjerrum-Bohr , Damgaard, Sondergaard; Mafra, Schlotterer , Stieberger

5) Construction of Lagrangians with duality and double copy properties, valid through 6 point trees. ZB, Dennen, Huang, Kiermaier

6) In self-dual case, identification of symmetry. Montiero and O'Connell