SUSY Gauge Theories
and
Quantization of Integrable Systems

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Based on:

- arXiv:0908.4052, arXiv:0901.4748, arXiv:0901.4744 (N. Nekrasov, S. Sh.)
- A. Gerasimov, S. Sh., '07, '08;
- G. Moore, N. Nekrasov, S. Sh. '97

In the last 15 years it become clear that the gauge theory dynamics in the vacuum sector is related to quantum many-body systems.

Quantum many body system $\Leftrightarrow$ topological gauge theory $\Leftrightarrow$ supersymmetric vacuum sector of SUSY gauge theory.

A classic example:

The system of N free non-relativistic fermions on a circle

$$\Updownarrow$$

2d pure Yang-Mills theory with gauge group $U(N)$

$$\Updownarrow$$

SUSY vacuum sector of a (deformed) $\mathcal{N} = 2$ super-Yang-Mills theory in 2d (on cylinder) Witten '92

Also describes (Witten '92) the intersection theory on moduli space of flat connections on a 2d Riemann surface $\Sigma_g, F(A) = 0$.

$k \to \infty$ limit of Gauged WZW_k (Verlinde formula).

A bit more complicated example MNS '97; GS '07, '08:

N-particle Yang system on circle S^1

$\Updownarrow$

2d YMH theory with gauge group $U(N)$

$\Updownarrow$

SUSY vacuum sector of a (deformed) 2d $\mathcal{N}=2$ theory, softly broken $\mathcal{N}=4$, $U(N)$ theory with massive adjoint matter (on cylinder)

Yang system - N-particle sector for the quantum Nonlinear Schrödinger equation (NLS), N non-relativistic particles on S^1:

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial {x_i}^2} + c \sum_{i \neq j} \delta(x_i - x_j)$$

YMH describes the $U(1)$-equivariant intersection theory on the moduli space of solutions to Hitchin's equations on Σ_g:

$$F_{z\bar{z}}(A) - [\Phi_z, \Phi_{\bar{z}}] = 0;$$

$$\nabla_z(A)\Phi_{\bar{z}} = 0; \quad \nabla_{\bar{z}}(A)\Phi_z = 0$$

$U(1)$ action:

$$\Phi_z \to e^{i\alpha}\Phi_z; \qquad \Phi_{\bar{z}}, \to e^{-i\alpha}\Phi_{\bar{z}}$$

Space of solutions, modulo gauge transformations, is isomorphic to

$$F_{z\bar{z}}(A + i\Phi) = 0$$

modulo complexified gauge transformations G^C.

Equivariant parameter enters as invariant mass term $\mathcal{L}_c = -c\Phi_z\Phi_{\bar{z}}$ and for $c \to \infty$ gives previous example - free fermion point.

$k \to \infty$ limit of Gauged WZW_k, for G^C.

The correspondence turns out to be much more general (NS '08):

For every quantum integrable system, solved by BA, there is a SUSY gauge theory with 4 supercharges, $Q_+, Q_-, \bar{Q}_+, \bar{Q}_-$, s.t.:

a) exact Bethe eigenstates correspond to SUSY vacua,

b) ring of commuting Hamiltonians $\Leftrightarrow$ (twisted) chiral ring.

SUSY vacuum equations in gauge theory $\Leftrightarrow$ Bethe equations

- Vacua: "critical" pts of effective twisted superpotential $\tilde{W}^{eff}(\sigma)$
- Bethe equations: spectrum, critical points of Yang function $Y(\lambda)$
- *The effective twisted superpotential corresponds to Yang function*

$$\boxed{\tilde{W}^{eff}(\sigma) = Y(\lambda)}$$

$$\sigma_i = \lambda_i; \quad i = 1, ..., N; \quad G = U(N)$$

- VEV of chiral ring operators O_k $\Leftrightarrow$ eigenvalues of Hamiltonians:

$$\boxed{< \lambda|O_k|\lambda >= E_k(\lambda)}$$

$$H_k \Psi(\lambda) = E_k(\lambda)\Psi(\lambda)$$

$\tilde{W}^{eff}(\sigma)$ - effective twisted superpotential on Couloumb branch as function of abelian components of scalar field σ_i
$Y(\lambda)$ - Yang's function as a function of rapidities λ_i

Explicit details worked out = gauge theories identified, for:

- XXX spin chain - 2d gauge theory
- XXZ spin chain - 3d gauge theory on $R^2 \times S^1$
- XYZ spin chain - 4d gauge theory on $R^2 \times T^2$
- Arbitrary spin group, representation, impurities, limiting models
- NLS, Yang system of N-particles on S^1 - 2d $\mathcal{N} = 4$ +...

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IN THIS TALK WE FOCUS ON (NS '09)

- Periodic Toda - 4d pure $\mathcal{N} = 2$ theory on $R^2 \times R^2_\epsilon$
- Elliptic Calogero-Moser - 4d $\mathcal{N} = 2^*$ theory on $R^2 \times R^2_\epsilon$

$\mathcal{N} = 2$ in $2d$

$\mathcal{N} = 2$ supersymmetry algebra in 2d has four generators which are the components of the two Dirac spinors: $Q_+, \quad Q_-, \quad \bar{Q}_+, \quad \bar{Q}_-$

$$\{Q_\pm, \bar{Q}_\pm\} = 2(H \pm P)$$

$$Q_\pm^2 = 0, \qquad \bar{Q}_\pm^2 = 0$$

$$Q_+^\dagger = \bar{Q}_+, \qquad Q_-^\dagger = \bar{Q}_-$$

The basic super-multiplets depend on space-time coordinates x, and 4 anti-commuting θ's ($\theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-$), some Reps of G:

$\mathbf{X}$ - chiral multiplets, matter - (charged) complex scalar:

$\mathbf{V}$ - vector multiplet: gauge field, complex adjoint scalar σ

$\mathbf{\Sigma}$ - the twisted chiral multiplets: σ, gauge field strength F_{01}

$$\mathbf{\Sigma} = \mathcal{D}_+ \bar{\mathcal{D}}_- V$$

SUSY Lagrangian: sum of $D, F, \tilde{F}, \tilde{m}$ and θ terms:

$$D: \qquad \int d^2x\, d^4\theta \left(-\frac{1}{4e^2}\, \mathrm{tr}\, \mathbf{\Sigma}\bar{\mathbf{\Sigma}} + K(e^{\mathbf{V}/2}\, \mathbf{X}\, ,\, \bar{\mathbf{X}}\, e^{\mathbf{V}/2}) \right)$$

$$F: \qquad \int d^2x\, d\theta^+ d\theta^- \; W(\mathbf{X})$$

$$\tilde{F}: \qquad \int d^2x\, d\theta^+ d\bar{\theta}^- \; \tilde{W}(\mathbf{\Sigma})$$

Twisted mass, $\tilde{m}$, term: suppose $\mathbf{X} \in \mathbf{R} = \oplus_{\bar{\imath}}\, \mathbf{M}_{\bar{\imath}} \otimes \mathbf{R}_{\bar{\imath}}$ - Global Symmetry group $H \subset \times_{\bar{\imath}}\, U(M_{\bar{\imath}})$

$$\tilde{m}: \qquad \int d^4\theta \;\; \mathrm{tr}_{\, R}\, \mathbf{X}^\dagger \left(\sum_{\bar{\imath}} e^{\tilde{V}_{\bar{\imath}}} \otimes \mathrm{Id}_{R_{\bar{\imath}}} \right) \mathbf{X}$$

$$\tilde{V}_{\bar{\imath}} = \tilde{m}_{\bar{\imath}}\, \theta_+ \bar{\theta}_- + c.c.$$

$\tilde{m}$ - belong to the complexification of the Lie algebra of the maximal torus of H: $\tilde{m} = (\tilde{m}_{\bar{\imath}})\;, \tilde{m}_{\bar{\imath}} \in \mathrm{End}\,(M_{\bar{\imath}})\,, [\tilde{m}_{\bar{\imath}}, \tilde{m}^*_{\bar{\imath}}] = 0$.
Need some unbroken global group H, $W(X)$ must preserve it.

θ-term - for each $U(1)$ component of gauge group $\theta_{\mathbf{a}} \int trF^{\mathbf{a}}$.

Promote complexified θ-term (which includes FI-term r) to Σ':

$$\theta : \qquad \int d^2x d\theta^+ \bar{\theta}^- [\Sigma'_{\mathbf{a}} \, \mathrm{tr} \, \Sigma^{\mathbf{a}}]$$

$$\Sigma'_{\mathbf{a}} = t_{\mathbf{a}} + ... = (\frac{\theta_{\mathbf{a}}}{2\pi} + ir_{\mathbf{a}}) + ...$$

Generic twisted masses - all matter fields are massive, can be integrated out and get $\mathcal{N} = 2$ gauge theory with an infinite number of interaction terms in the Lagrangian; high derivative terms suppressed by the inverse masses of the fields we integrated out.

In addition there are other massive fields which can be integrated out on the Coulomb branch. These are the $\mathbf{g}/\mathbf{t}$-components ($\mathbf{g}$ - Lie algebra corresponding to Lie group G, $\mathbf{t}$ - its Cartan sub-algebra) of the vector multiplets, the W-bosons ...

Twisted F-term, the $\tilde{F}$-term $\tilde{W}^{\text{eff}}(\Sigma) \Leftrightarrow \tilde{W}^{\text{eff}}(\sigma + \ldots)$, can be computed exactly - receives only one-loop contributions.

Effective theory is abelian with field content of pure $\mathcal{N} = 2$:

$$\tilde{W}^{\text{eff}}(\sigma) = \tilde{W}^{\text{eff}}_{\text{matter}}(\sigma) + \tilde{W}^{\text{eff}}_{\text{gauge}}(\sigma)$$

$$\tilde{W}^{\text{eff}}(\sigma) = -\sum_{\mathbf{b}} 2\pi i t_{\mathbf{b}} tr_{\mathbf{b}} \sigma + tr_R\left(\sigma + \tilde{\mathbf{m}}\right)\left(\log\left(\sigma + \tilde{\mathbf{m}}\right) - 1\right) -$$

$$- 2\pi < \rho, \sigma >, \qquad \mathbf{t_b} = \frac{\theta_{\mathbf{b}}}{2\pi} + i r_{\mathbf{b}}, \quad \rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$$

- Only $d\tilde{W}^{\text{eff}}(\sigma)$ enters in effective Lagrangian $\mathcal{L}$.
- One can consider a 3d and 4d supersymmetric gauge theories which when reduced to 2d on S^1 and T^2 give above 2d theories.
- Again, one can write the explicit formula for effective twisted superpotential in 2d - including the contribution of all KK-modes.

Supersymmetric vacua of $\mathcal{N}=2$ theories

For any $\mathcal{N}=2$ theory we can write Hamiltonian as (in the absence of central extentions):

$$\{Q_A, Q_A^\dagger\} = \{Q_B, Q_B^\dagger\} = 4H$$

$$Q_A = Q_+ + \bar{Q}_-; \qquad Q_A^\dagger = Q_- + \bar{Q}_+; \quad Q_A^2 = 0$$

$$Q_B = Q_+ + Q_-; \qquad Q_B^\dagger = \bar{Q}_+ + \bar{Q}_-; \quad Q_B^2 = 0$$

SUSY vacua are annihilated by H:

$$H|0>= 0$$

First address a simpler question - Q_A (Q_B)-cohomology:

$$Q_{A(B)}|\Psi>= 0; \quad |\Psi>\sim |\Psi> + Q_{A(B)}|\ldots>$$

Vacuum state is a “harmonic” representative in this cohomology.

If $|0>$ is some state in vacuum and O_i is in Q cohomology

$$\{Q, O_i\} = 0, \qquad O_i \sim O_i + \{Q, ...\}$$

$|i> = O_i|0>$ is also a vacuum state.

Operator-state correspondence would relate the complete basis for vacuum states $|i>$ to operators from cohomology O_i.

• These operators are independent of position up to Q-comm.

$$dO_i = \{Q, ...\}$$

• They form a commutative ring called chiral ring:

$$O_i O_j |0> = c_{ij}^k O_k |0>; \quad \Rightarrow \quad O_i O_j = c_{ij}^k O_k + \{Q, ...\}$$

• In good situation (mass gap) chiral ring generators can be written in terms of Coulomb branch - $O_k = tr\sigma^k$.

• SUSY vacua form the representation of chiral ring.

Basically, for every $\mathcal{N} = 2$ theory there is a quantum integrable system (assuming all good conditions - discrete specturm ...).

What are these quantum integrable systems?

After all massive fields are integrated out chiral ring generators are invariant functions on Coulomb branch, functions of $\Sigma = \sigma + \ldots$.

SUSY vacua - there are two options: 1. topological or 2. physical.

1. Topologically twisted (on cylinder) abelianized theory has the action completely determined by $\tilde{W}^{eff}(\sigma)$ of physical theory:

$$S_{top} = \int \left[\frac{\partial \tilde{W}^{eff}(\sigma)}{\partial \sigma_i} F^i(A) + \frac{\partial^2 \tilde{W}^{eff}(\sigma)}{\partial \sigma_i \partial \sigma_j} \lambda^i \wedge \lambda^j \right]$$

$$compare \qquad S_{2d-YM} = \int \left[\sigma_i F^i(A) + \lambda^i \wedge \lambda^j \right]$$

Canonical quantization - momentum conjugate to the monodromy of abelian gauge field $x^i = \int_{S^1} A^i$ is quantized:

$$\frac{1}{2\pi i} \frac{\partial \tilde{W}^{eff}(\sigma)}{\partial \sigma^i} = n_i$$

2. Physical: suppose we have the theory with the effective twisted superpotential $\tilde{W}^{\text{eff}}(\sigma)$ (abelianized).

The target space of the effective sigma model is disconnected, with $\vec{n}$ labeling the connected components (gauge flux quantization) with potential:

$$U_{\vec{n}}(\sigma) = \frac{1}{2} \mathrm{g}^{ij} \left(-2\pi i n_i + \frac{\partial \tilde{W}^{\text{eff}}}{\partial \sigma^i} \right) \left(+2\pi i n_j + \frac{\partial \bar{\tilde{W}}^{\text{eff}}}{\partial \bar{\sigma}^j} \right)$$

Now we need to find the minimum of potential - again:

$$\frac{1}{2\pi i} \frac{\partial \tilde{W}^{\text{eff}}(\sigma)}{\partial \sigma^i} = n_i$$

Or equivalently - SUSY vacua correspond to solution of equation:

$$\exp\left(\frac{\partial \tilde{W}^{\text{eff}}(\sigma)}{\partial \sigma^i} \right) = 1$$

$\underline{The}$ $\underline{Main}$ $\underline{example}$: $\tilde{Q}\Phi Q$ $\underline{theory}$ - XXX_s

Gauge group $G = U(N)$

L fundamental chiral multiplets $\mathbf{Q}_a$,

L anti-fundamental chiral multiplets $\tilde{\mathbf{Q}}^a$

One adjoint chiral multiplet $\mathbf{\Phi}$.

This matter content corresponds to the gauge theory with extended supersymmetry, $\mathcal{N} = 4$, which the dimensional reduction of the four dimensional $\mathcal{N} = 2$ theory.

The adjoint $\mathbf{\Phi}$ is a part of the vector multiplet in $4d$, while chiral fundamental and anti-fundamentals combine into hypermultiplet in the fundamental representation. We are dealing, therefore, with the matter content of the four dimensional $\mathcal{N} = 2$ theory with $N_c = N$, $N_f = L$.

Gauge group has a center $U(1)$ - turn on the FI term and the theta angle, combine into a complexified coupling $\theta \mapsto t = \frac{\theta}{2\pi} + ir$.

m_a^{f} - twisted mass for the fundamental chiral fields Q_a,

$m_a^{\bar{\mathrm{f}}}$ - the twisted masses for the anti-fundamental chiral fields $\tilde{Q}^a$,

m^{adj} the twisted masses for the adjoint Φ.

$$\tilde{W}_{\tilde{Q}\Phi Q} = \sum_{i=1}^{N} \sum_{a=1}^{L} [\left(\sigma_i + m_a^{\mathrm{f}}\right) \left(\log\left(\sigma_i + m_a^{\mathrm{f}}\right) - 1\right) +$$

$$+ \left(-\sigma_i + m_a^{\bar{\mathrm{f}}}\right) \left(\log\left(-\sigma_i + m_a^{\bar{\mathrm{f}}}\right) - 1\right)] +$$

$$+ \sum_{i,j=1}^{N} \left(\sigma_i - \sigma_j + m^{\mathrm{adj}}\right) \left(\log\left(\sigma_i - \sigma_j + m^{\mathrm{adj}}\right) - 1\right) -$$

$$- 2\pi i \sum_{i=1}^{N} \left(t + i - \frac{1}{2}(N+1)\right) \sigma_i$$

$$\prod_{a=1}^{L} \frac{\sigma_i + m_a^{\mathrm{f}}}{\sigma_i - m_a^{\bar{\mathrm{f}}}} = -e^{2\pi i t} \prod_{j=1}^{N} \frac{\sigma_i - \sigma_j + m^{\mathrm{adj}}}{\sigma_i - \sigma_j - m^{\mathrm{adj}}}$$

Same equation in invariant form, in terms of $\mathbf{Q}(x) = det(x - \sigma)$

$$a(x)\mathbf{Q}(x - m^{\mathrm{adj}}) + e^{2\pi i t} d(x)\mathbf{Q}(x + m^{\mathrm{adj}}) = t(x)\mathbf{Q}(x)$$

$$a(x) = \prod_{a=1}^{L}(x - m_a^{\bar{\mathrm{f}}}); \quad d(x) = \prod_{a=1}^{L}(x + m_a^{\mathrm{f}})$$

$t(x)$ - an unknown polynomial of degree L, coeff. - chiral ring

- Turn on $W(X) = \sum_a \varpi_a \tilde{Q}^a \Phi^{2s_a} Q_a$, global symmetry restricts:

$$m_a^{\mathrm{f}} = -\mu_a - i s_a u, \quad m_a^{\bar{\mathrm{f}}} = +\mu_a - i s_a u, \quad m^{\mathrm{adj}} = iu, \quad s_a \in \frac{1}{2} Z$$

- BA of periodic, length L, XXX_s in "N-particle" sector: μ_a's - impurities, s_a's- spins, $\tilde{W}$ - Yang-function.
- Invariant form: vacuum Ward Identity - Baxter equation.

4d SYM and Algebraic Integrable Systems

Low energy effective action in $U(N)$ 4d pure $\mathcal{N}=2$ SYM is described by prepotential $\mathcal{F}(a;\Lambda)$, SW '94; has interpretation in terms of classical algebraic integrable system - Periodic Toda.

Completely integrable classical Hamiltonian system - $2r$-dimensional symplectic manifold (M,Ω_R,H_i) which posses r independent mutually commuting functions $H_i \in R^r$: $\{H_i,H_j\}=0$. H_i define Lagrangian fibration $H: M \to B \in R^r$.

If common level set $H^{-1}(h)$ is compact - it is diffeomorphic to T^r. Locally trivial - isomorphic to $T^r \times \mathcal{U}$, $\mathcal{U} \in B$.

One can define special Darboux coordinates, action-angle: $\Omega_R = \sum_{i=1}^r dI^i \wedge d\varphi_i$. φ_i - periodic angular variables on fibers T^r. Let γ_i be a Z bases in $H_1(T^r_S, Z)$ smooth in $s \in B$:

$$I^i = \int_{\gamma_i} \Theta, \qquad \Omega_R(x) = d\Theta(x), \qquad x \in H^{-1}(\mathcal{U})$$

Algebraic Completely Integrable System (M, Ω_C, H_i):

- A complex algebraic manifold M of complex dimension $2r$
- Everywhere non-degenerate, closed holomorphic (2, 0)-form $\Omega_C^{2,0}$
- A holomorphic map $H : M \to C^r$, fibers $J_h = H^{-1}(h)$ are (polarized) abelian varieties (complex tori), $\{H_i, H_j\} = 0$

Polarization is a Kahler form ω whose restriction on each fiber is integral class: $[w] \in H^2(J_h, Z) \cap H^{1,1}(J_h)$

We have twice as many "action variables" - thus they are related. Again, restricted to fibers:

$$\Omega_C = d\Theta_C$$

$$a_i = \int_{A_i} \Theta_C, \qquad a_D^i = \int_{B^i} \Theta_C,$$

over the A and B-cycles (dual $< A_i, B^j >= \delta_i^j$, which form bases in $H_1(J_h, Z)$), Lagrangian with respect to the intersection form ω.

There should be only r independent action variables. Locally:

$$a_D^i = \frac{\partial \mathcal{F}(a)}{\partial a^i}$$

One can chose $\{a_D^j\}$ as action variables:

$$\Omega_C = \sum_{i=1}^{r} da_D^i \wedge d\varphi_D^i$$

The base can be supplied with the Rigid Special Geometry structure - locally a Lagrangian submanifold (holomorphic) in C^{2r}:

$$\Omega = \sum_{j=1}^{r} da_j \wedge da_D^j$$

$$d^2 s = \sum_{j=1}^{r} da_j \otimes d\bar{a}_D^j - da_D^j \otimes d\bar{a}_j$$

Restriction of one form $\theta = \sum_i a_D^i da_i$ to Lagrangian submanifold is exact and defines embeding via pre-potential $\mathcal{F}(a)$:

$$\theta = \sum_i a_D^i da_i = d\mathcal{F}(a)$$

$\mathcal{N} = 2^*$ and Elliptic Calogero-Moser

$U(N)$ 4d $\mathcal{N} = 2^*$ theory is the $\mathcal{N} = 2$ theory with massive adjoint hypermultiplet; coupling constant - $\tau = \frac{i}{g_0^2} + \theta_0$, mass - m.

Low energy effective theory is described in terms of prepotential $\mathcal{F}(a_1, ..., a_N; \tau, m)$ which comes from Elliptic Calogero-Moser (eCM) algebraic completely integrable system.

eCM - N particles $q_1, q_2, ..., q_N$ on the circle of circumference β , $q_i \sim q_i + \beta$, which interact with the pair-wise potential:

$$H_2 = \sum_{i=1}^{N} p_i^2 + U(q); \qquad U(q) = m^2 \sum_{i<j} \mathcal{P}(q_i - q_j)$$

$$\mathcal{P}(x) = \sum_{n \in Z} \frac{1}{sinh^2(x + n\beta)} = u_0(x) + \sum_{k=1}^{\infty} q^k u_n(x)$$

$$q = e^{-2\beta}; \quad u_0 = \frac{1}{sinh^2 x} = \sum_k n e^{-kx}; \quad u_k(x) = 4 \sum_{d|k} d(e^{dx} + e^{-dx})$$

Introduce the Lax operator $\Phi(z|p,q)$ - $N \times N$ matrix,

$$\Phi_{ij}(z|p,q) = p_i\delta_{ij} + m\frac{\Theta(z+q_i-q_j)\Theta'(0)}{\Theta(q_i-q_j)\Theta(z)}(1-\delta_{ij})$$

$$\Theta(x) = -\sum_{k=Z+\frac{1}{2}}(-1)^k q^{\frac{k^2}{2}} e^{2kx}; \quad q = e^{2\pi i\tau}; \quad \tau = \frac{i\beta}{\pi}$$

and define all Hamiltonians H_k as coefficients in front of x^k of characteristic polynomial: $\det(\Phi(z)-x)$

Spectral curve: $\mathcal{C}_h \subset C \times C^{\times}$ is defined as zero locus of characteristic polynomial:

$$\det(\Phi(z)-x) = 0$$

$H^{-1}(h)$ is given by the product $C \times J_h$. The C-factor corresponds to the center-of-mass mode $\sum_i q_i$, while the compact factor $J_h = Jac(\bar{C}_h)$ is the Jacobian of the compactied curve C_h .

a_i, a_D^i are periods of differential $\lambda = \frac{1}{2\pi} x dz$. Corresponding $\mathcal{F}(a)$ gives prepotential for $\mathcal{N} = 2^*$ theory.

In the limit $\beta \to \infty (q \to 0), m \to \infty$ with $\Lambda^{2N} = m^{2N} q$ finite - eCM $\to$ pToda.

$$H_2 = \sum_{i=1}^{N} p_i^2 + U(q); \qquad U(q) = \Lambda^2 (\sum_{i=1}^{N-1} e^{q_i - q_{i-1}} + e^{q_N - q_1})$$

In the same limit - $\mathcal{N} = 2^*$ theory becomes pure $\mathcal{N} = 2$ SYM.

$$\mathcal{F}(a) = \mathcal{F}^{pert}(a) + \mathcal{F}^{non-pert}(a)$$

$$\mathcal{F}^{pert}(a; \tau, m) = \frac{\tau}{2} \sum_{i=1}^{N} a_i^2 + \frac{3N^2 m^2}{2} + \frac{1}{4} \sum_{i,j=1}^{N} [(a_i - a_j)^2 log(a_i - a_j) - \\ - (a_i - a_j + m)^2 log(a_i - a_j + m)]$$

$$\mathcal{F}^{non-pert}(a; \tau, m) = \sum_{k=1}^{\infty} q^k \mathcal{F}_k(a; m), \quad q = e^{2\pi i \tau}$$

Quantization $\Leftrightarrow$ Deformation of SYM

Suppose we quantize the algebraic integrable system (ϵ - Planck).
If we chose a_i^D as our action variables than Bohr-Sommerfeld:

$$a_i^D = \epsilon \times n_i = \frac{\partial \mathcal{F}(a)}{\partial a_i}$$

$$\frac{\partial Y(a)}{\partial a_i} = n_i; \qquad Y(a) = \frac{\mathcal{F}(a)}{\epsilon}$$

This semi-classical picture is very suggestive - Bethe equation:

$$\frac{\partial Y(a;\epsilon)}{\partial a_i} = n_i$$

$Y(a;\epsilon)$ - Yang function.

We look for quantization when $Y(a,\epsilon)$ has a Laurent series expansion in ϵ starting with a single pole and residue $\mathcal{F}(a)$

$$Y(a;\epsilon) = \frac{\mathcal{F}(a) + O(\epsilon)}{\epsilon}$$

Appearance of prepotential in $\mathcal{N} = 2$ and $\mathcal{N} = 2^*$ suggests, from our experience in gauge theory $\Leftrightarrow$ quantum integrablity: we start with this 4d SYM, deform it in ϵ and count the vacua.

These theories have continues spectrum of vacua - "u"-plane.

In order to find discrete spectrum and $Y(a;\epsilon)$ - we need ϵ deformation of 4d theory such that the low energy theory is:

2d with superpotential: $\mathcal{W}(a,\epsilon) = Y(a;\epsilon) = \frac{\mathcal{F}(a)}{\epsilon} + ...$

In fact we know such theory - 4d gauge theory on $R^2 \times R^2_\epsilon$.

2d character of low energy action is best explained, and computed exactly, in terms of topological gauge theory, which always explains the vacuum sector precisely.

Our main examples - pToda (pure $\mathcal{N}=2$) and eCM ($\mathcal{N}=2^*$).

$\mathcal{N}=2$ gauge theory on $R^2 \times R^2_\epsilon$ is a deformation of $\mathcal{N}=2$ theory on $R^2 \times R^2$ with one, equivariant, parameter ϵ which corresponds to the rotation of second R^2 around its origin.

Denote corresponding vector field $V = \epsilon(x^2\partial_3 - x^3\partial_2)$.

Bosonic part is:

$$L = \frac{1}{g_0^2}(-\frac{1}{2}trF \star F + Tr(D_A\phi - i_V F) \star (D_A\bar{\phi} - i_{\bar{V}}F) +$$

$$+ \frac{1}{2}Tr([\phi, \bar{\phi}] + i_V D_A\bar{\phi} - i_{\bar{V}} D_A\phi)^2 + \frac{\theta_0}{2\pi} TrF \wedge F$$

Only 2d (first R^2) super-Poincare invariance is unbroken, four Q's. Alternative to KK - 2d theory with infinite number of fields in UV.

This theory has twisted formulation (together with deformation by chiral ring operators with $\{\mathbf{t}\} = (t_1, ..., t_N)$, $LNS \quad '97$), ϵ-def. of Donaldson-Witten. Its abelianization (effective low energy) is 2d gauge theory with four Q's and superpotential (for $\mathcal{N} = 2^*$):

$$W(a|\{\mathbf{t}\}; m, \epsilon, \tau) = \frac{\mathcal{F}(a|\{\mathbf{t}\}; m, \tau) + O(\epsilon)}{\epsilon}$$

W is sum of perturbative in τ and non-pertubative (infinite series in $q = e^{2\pi i \tau}$ for $\mathcal{N} = 2^*$) - known exactly as:

$$W = W_{pert}(a|\{\mathbf{t}\}; m, \epsilon, \tau) + \sum_{k=1}^{\infty} q^k W_k(a|\{\mathbf{t}\}; m, \epsilon)$$

Second part can be computed using the integral formulas of MNS for W_k or integral equation representation for $W_{inst} = \sum_k q^k W_k$.

What is exactly the quantization problem for which this W gives the Yang function and SUSY vacua - the exact spectrum?

$$\frac{\partial W(a|\{\mathbf{t}\}, \epsilon, \tau)}{\partial a_i} = n_i$$

• For eCM replace $p_i = \epsilon \frac{\partial}{\partial q_i}$, and q_i, m^2, ϵ - complex

• Write the eigenvalue problem for all Hamiltonians, parametrize eigenvalues $E_1, ..., E_N$ in terms of $a_1, ..., a_N$ - e. g. for H_2:

$$[\frac{\epsilon^2}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial {q_i}^2} + m(m+\epsilon) \sum_{i<j} \mathcal{P}(q_i - q_j; \beta)] \Psi(q) = E_2(a)\Psi(q)$$

$$\epsilon = -i\hbar, \quad m = i\hbar\nu \quad \Rightarrow \quad m(m+\epsilon) = -\hbar^2 \nu(\nu - 1)$$

• Look for solutions in affine Weyl chamber with asympthotics at $(q_i - q_j) \to 0$ of $\Psi \to (q_i - q_j)^\nu$, and extend outside this domain by symmetry condition with respect to shift in β.

• Spectrum ($q = e^{2\pi i\tau} = e^{-2\beta}$):

$$\frac{\partial W_{\mathcal{N}=2^*}(a|\{\mathbf{t}\}; m, \epsilon, \tau)}{\partial a_i} = n_i; \quad E_i(a) = \frac{\partial W_{\mathcal{N}=2^*}(a|\{\mathbf{t}\}; m, \epsilon, \tau)}{\partial t_i}$$

Checked in q-expansion knowing $W(a|\{\mathbf{t}\}, m, \epsilon, \tau)$ for $\mathcal{N} = 2^*$.

$$E_2 = \epsilon q \frac{\partial}{\partial q} W_{\mathcal{N}=2^*}(a|\{\mathbf{t}=0\}; m, \epsilon, \tau)$$

As already explained, the vacuum equation is:

$$\exp(\frac{\partial W(a|\{\mathbf{t}\};m,\epsilon,\tau)}{\partial a_i}) = 1$$

$W = W_{pert} + W_{inst}$ and the perturbative part of equation has simple form for $t = 0$:

$$1 = \exp(\frac{\partial W_{\text{pert}}(a|\{\mathbf{t}=0\},m,\epsilon,\tau)}{\partial a_i}) = e^{\frac{\pi i \tau a_i}{\epsilon}} \prod_{j\neq i} S(a_i - a_j)$$

$$S(x) = \frac{\Gamma\left(\frac{-m+x}{\epsilon}\right)}{\Gamma\left(\frac{-m-x}{\epsilon}\right)} \frac{\Gamma\left(1-\frac{x}{\epsilon}\right)}{\Gamma\left(1+\frac{x}{\epsilon}\right)}$$

$$W(a;q,m,\epsilon) = W_{pert}(a;q,m,\epsilon) + \mathrm{Limit}_{\epsilon_2 \to 0}\epsilon_2 \log \mathcal{Z}_{inst}(a;q,m,\epsilon_1,\epsilon_2)$$

$$\mathcal{Z}^{\mathrm{inst}}(a;q,m,\epsilon_1,\epsilon_2) =$$

$$\sum_{k=0}^{\infty} \frac{q^k}{k!} \int_k \prod_{1 \le I < J \le k} \frac{R_+(\phi_{IJ})}{R_-(\phi_{IJ})} \prod_{I=1}^{k} Q(\phi_I) \frac{\epsilon\,(m+\epsilon_1)(m+\epsilon_2)}{\epsilon_1\epsilon_2\, m(m+\epsilon)} \frac{\mathrm{d}\phi_I}{2\pi i}$$

$$\epsilon = \epsilon_1 + \epsilon_2; \quad \phi_{IJ} = \phi_I - \phi_J$$

$$R_+(x) = x^2(x^2-\epsilon^2)(x^2-(m+\epsilon_1)^2)(x^2-(m+\epsilon_2)^2)$$

$$R_-(x) = (x^2-\epsilon_1^2)(x^2-\epsilon_2^2)(x^2-m^2)(x^2-(m+\epsilon)^2)$$

$$Q(x) = \frac{P(x-m)P(x+m+\epsilon)}{P(x)P(x+\epsilon)}; \quad P(x) = \prod_{l=1}^{N}(x-a_l)$$

Solve the integral (non-linear) equation:

$$\chi(x) = \int \mathrm{d}y\, G_0(x-y) \log\left(1 - q e^{-\chi(y)} Q(y)\right)$$

$$G_0(x) = \partial_x \log \frac{(x+\epsilon)(x+m)(x-m-\epsilon)}{(x-\epsilon)(x-m)(x+m+\epsilon)}$$

On solutions of this equation evaluate the functional:

$$W_{inst}(a) = \int \mathrm{d}x \left[-\frac{\chi(x)}{2} \log\left(1 - qQ(x)e^{-\chi(x)}\right) + \right.$$

$$\left. + \operatorname{Li}_2\left(qQ(x)e^{-\chi(x)}\right) \right]$$