Massive 3D (super)gravity

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Based on work with Olaf Hohm and Eric Bergshoeff, [BHT]
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Outline

- Spin 1 Warm Up: $\begin{cases} Proca \\ Generalized Proca \\ Self-Dual\ spin\ 1 \\ Extended\, Proca \end{cases}$

- Spin 2: $\begin{cases} (Generalized) Fierz-Pauli \\ Self-Dual\ spin\ 2 \\ TMG \\ NMG \begin{cases} Unitarity \\ Renormalizability? \end{cases} \\ GMG \\ NTMG \end{cases}$

- Unification via $\mathcal{N}=2$ Sugra

- Cosmological models : $\begin{cases} adS\ vacua \\ bulk\ unitarity\ bound \\ boundary\ CFT \end{cases}$

- Outlook

3D Proca

$$\mathcal{L}_{Proca} = -\tfrac{1}{4}G^{\mu\nu}G_{\mu\nu} - \tfrac{1}{2}m^2B^2\,, \qquad \left(G_{\mu\nu} = 2\partial_{[\mu}B_{\nu]}\right)$$

➽ Equations of motion are equivalent to

$$\left(\Box - m^2\right)B_\mu = 0\,, \qquad \partial^\mu B_\mu = 0 \qquad\qquad (\dagger)$$

Propagates $(D-1)$ modes in D spacetime dimensions, with mass m, so two modes for $D=3$.

➽ For $D=3$ the Proca equations are equivalent to

$$\left[\mathcal{P}(m)\mathcal{P}(-m)\right]_\mu{}^\rho B_\rho = 0 \qquad \mathcal{P}(m)_\mu{}^\nu = \tfrac{1}{2}\left[\delta_\mu^\nu - \tfrac{1}{m}\varepsilon_\mu{}^{\tau\nu}\partial_\tau\right]$$

➽ The operator $\mathcal{P}(m)$ is an on-shell projection operator

$$\mathcal{P}^2(m)B = \mathcal{P}(m)B \qquad \text{if } B \text{ satisfies } (\dagger)$$

It projects onto on-shell fields of helicity $h = \pm 1$:

$$\mathcal{P}(m) = \tfrac{1}{2}\left[1 - \mathrm{sgn}(m)h\right]\,, \qquad h = (P\cdot J)\,/|m|$$

$\Rightarrow$ two propagated modes have helicities $+1$ and -1

Generalized 3D Proca and 'self-dual' limit

$$\mathcal{L}_{GP} = \tfrac{1}{2}\tilde{G}^2 - \tfrac{1}{2}\tilde{\mu}\,\varepsilon^{\mu\nu\rho}B_\mu\partial_\nu B_\rho - \tfrac{1}{2}m^2B^2\,, \qquad \left(\tilde{G}^\mu = \varepsilon^{\mu\nu\rho}\partial_\nu B_\rho\right)$$

- Recover Proca in limit $\tilde{\mu} \to \infty$ for fixed m.
- For finite $\tilde{\mu}$ we have parity breaking Chern-Simons (CS) term
- Equations of motion are equivalent to

$$[\mathcal{P}(m_+)\mathcal{P}(-m_-)]_\mu{}^\rho B_\rho = 0\,, \qquad m_+m_- = m^2\,, \qquad m_- - m_+ = \tilde{\mu}$$

$\Rightarrow$ helicities ± 1 propagated with mass $m_\pm$

- Take $m_- \to \infty$ for fixed m_+ to get $\sqrt{\text{Proca}}$ (or "self-dual") equation [Pilch, van Nieuwenhuizen, PKT, '84]

$$\mathcal{P}(\mu)B = 0\,, \qquad \mu = m^2/\tilde{\mu}$$

$\Rightarrow$ only helicity $+1$ propagated, with mass μ. Equation can be derived from 1st order Lagrangian

$$\mathcal{L}_{SD1} = -\tfrac{1}{2}\varepsilon^{\mu\nu\rho}B_\mu\partial_\nu B_\rho - \tfrac{1}{2}\mu B^2$$

Topologically massive electrodynamics

➽ $\sqrt{\text{Proca}}$ equation implies subsidiary condition $\partial \cdot B = 0$. Solve this in terms of vector potential A:

$$B^\mu = \varepsilon^{\mu\nu\rho}\partial_\nu A_\rho \equiv \tilde{F}^\mu$$

Substitute into $\sqrt{\text{Proca}}$ equation to get gauge-invariant equation

$$P(\mu)\tilde{F} = 0 \qquad \left(\partial \cdot \tilde{F} \equiv 0\right)$$

This is field equation of topologically massive electrodynamics [Schonfeld '81, Deser, Jackiw & Templeton, '82]

$$\mathcal{L}_{TME} = \tfrac{1}{2}\tilde{F}^2 - \tfrac{1}{2}\mu\,\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho$$

➽ **By construction**, TME is on-shell equivalent to $\sqrt{\text{Proca}}$. Off-shell equivalence follows on choosing sign of TME action to avoid ghosts (obviously possible because only one mode propagated).

➽ Can also prove equivalence of TME to $\sqrt{\text{Proca}}$ via 'master action' [Deser & Jackiw, '84]

Extended 3D Electrodynamics

➽ Solve subsidiary condition $\partial \cdot B = 0$ for Generalized Proca. This gives the 'extended GP' (EGP) equation

$$[\mathcal{P}(m_+)\mathcal{P}(-m_-)]_\mu{}^\rho \tilde{F}_\rho = 0$$

➽ EGP Lagrangian is

$$L_{EGP} = L_{CS} + \frac{1}{\mu} L_{Max} + \frac{1}{m^2} L_{ECS}$$

where

$$L_{CS} = \tfrac{1}{2}\varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho\,, \qquad L_{Max} = \tfrac{1}{2}\tilde{F}^2\,, \qquad L_{ECS} = +\tfrac{1}{2}\varepsilon^{\mu\nu\rho}\tilde{F}_\mu \partial_\nu \tilde{F}_\rho$$

The 3rd order term L_{ECS} is the "extended Chern-Simons" term

➽ Despite on-shell equivalence to GP, *there is no off-shell equivalence*. In the EGP theory, one of the two helicity modes is a ghost (negative KE) [Deser & Jackiw, '99].

3D Fierz-Pauli

$$\mathcal{L}_{FP} = \tfrac{1}{2}\varphi^{\mu\nu}\mathcal{G}_{\mu\nu}{}^{\rho\sigma}\varphi_{\rho\sigma} - \tfrac{1}{4}m^2\left(\varphi^{\mu\nu}\varphi_{\mu\nu} - \varphi^2\right) \qquad (\varphi = \eta^{\mu\nu}\varphi_{\mu\nu})$$

- $\varphi_{\mu\nu}$ is symmetric tensor field, $\mathcal{G}$ is the 'Einstein operator'

$$\mathcal{G}_{\mu\nu}{}^{\rho\sigma} = -\tfrac{1}{2}\varepsilon_{(\mu}{}^{\tau\rho}\varepsilon_{\nu)}{}^{\eta\sigma}\partial_\tau\partial_\eta$$

- $G^{(\text{lin})}_{\mu\nu}(\varphi) = \mathcal{G}_{\mu\nu}{}^{\rho\sigma}\,\varphi_{\rho\sigma}$ is linearized Einstein tensor. The 'Einstein operator' $\mathcal{G}$ is second order in derivatives and self-adjoint. For example, in 3D Minkowski spacetime with signature (-,+,+)

- Equations of motion are equivalent to

$$\left(\Box - m^2\right)\varphi_{\mu\nu} = 0\,, \qquad \partial^\mu\varphi_{\mu\nu} = 0\,, \qquad \varphi = 0$$

Propagates $(D+1)(D-2)/2$ modes with mass m, so two modes for $D = 3$.

- For $D = 3$ the FP equations are equivalent to

$$\left[\mathcal{P}(m)\mathcal{P}(-m)\right]_\mu{}^\rho\varphi_{\rho\nu} = 0\,, \qquad \varphi = 0$$

$\Rightarrow$ subsidiary condition $\partial^\mu\varphi_{\mu\nu} = 0$ $\Rightarrow$ dynamical eq. symmetric under $\mu \leftrightarrow \nu$ $\Rightarrow$ $\mathcal{P}(\pm m)$ projects onto helicity ± 1 for each vector index $\Rightarrow$ propagated modes have helicities $+2$ and -2.

Generalized 3D Fierz-Pauli

- In 3D we can replace FP equations by

$$[\mathcal{P}(m_+)\mathcal{P}(-m_-)]_\mu{}^\rho \varphi_{\rho\nu} = 0\,, \qquad \varphi = 0$$

These equations propagate one mode of helicity $+2$ with mass m_+ and another mode of helicity -2 with mass m_-.

- Take limit $m_- \to \infty$ for fixed m_+ to get $\sqrt{FP}$, or "self-dual" spin 2, equation [Aragone & Khoudeir, '86]

$$[\mathcal{P}(\mu)]_\mu{}^\rho \varphi_{\rho\nu} = 0\,, \qquad \varphi = 0$$

This propagates a single helicity 2 mode of mass μ

- $\sqrt{FP}$ equations imply subsidiary condition

$$\partial^\mu \varphi_{\mu\nu} = 0$$

- To get Lagrangian for either GFP or $\sqrt{FP}$, need to start from a 2nd rank tensor field of *no symmetry*. The equations of motion put the antisymmetric part to zero [Aragone & Khoudeir, '86]

Linearized TMG

➽ Solve differential subsidiary constraint of $\sqrt{FP}$ for $\varphi_{\mu\nu}$ in terms of a symmetric tensor potential $h_{\mu\nu}$:

$$\varphi^{\mu\nu} = -\tfrac{1}{2}\varepsilon^{\mu\tau\rho}\varepsilon^{\nu\eta\sigma}\partial_\tau\partial_\eta\, h_{\rho\sigma} \equiv G^{(\text{lin})}_{\mu\nu}(h)$$

Remaining $\sqrt{FP}$ equations become

$$[\mathcal{P}(\mu)]_\mu{}^\rho G^{(\text{lin})}_{\rho\nu}(h) = 0\,, \qquad R^{(\text{lin})}(h) = 0$$

where $R^{(\text{lin})} = -2\eta^{\mu\nu}G^{(\text{lin})}_{\mu\nu}$ is the linearized Ricci scalar.

➽ These equations can be derived from the Lagrangian

$$\mathcal{L}^{(\text{lin})}_{TMG} = \tfrac{1}{2}h^{\mu\nu}G^{(\text{lin})}_{\rho\nu} + \tfrac{1}{2\mu}h^{\mu\nu}C^{(\text{lin})}_{\mu\nu} \qquad (\star)$$

where

$$C^{(\text{lin})}_{\mu\nu} = \varepsilon_\mu{}^{\tau\rho}\partial_\tau S^{(\text{lin})}_{\rho\nu}\,, \qquad S^{(\text{lin})}_{\mu\nu} = R^{(\text{lin})}_{\mu\nu} - \tfrac{1}{4}\eta_{\mu\nu}R^{(\text{lin})}$$

This is the linearized Cotton tensor (3D analog of Weyl tensor)

➽ We now have 3rd order field equations, without ghosts (since equivalent on-shell to $\sqrt{FP}$ and we have chosen sign such that the one propagated helicity 2 mode is physical)

Topologically Massive Gravity

➽ Lagrangian $(\star)$ is quadratic approximation to Lagrangian of TMG [Deser, Jackiw & Templeton, '82]

$$\mathcal{L}_{TMG} = -\sqrt{|g|}R + \frac{1}{\mu}\mathcal{L}_{LCS}$$

The "Lorentz Chern-Simons" term is the CS term for the Levi-Civita connection, hence 3rd order in derivatives.

➽ Note '**wrong sign**' of Einstein-Hilbert term

➽ TMG field equations are

$$G_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0$$

➽ The Cotton tensor $C_{\mu\nu}$ is 3D analog of Weyl tensor: $C_{\mu\nu} \equiv 0$ implies 3-metric is conformally flat. Because of the identity $g^{\mu\nu}C_{\mu\nu} \equiv 0$, the TMG field equations imply that $R = 0$.

New Massive Gravity

➽ Solve differential subsidiary constraint of FP. Remaining FP equations become

$$\left(\Box - m^2\right) G^{(\text{lin})}_{\rho\nu} = 0\,, \qquad R^{(\text{lin})} = 0$$

These are equations of Lagrangian

$$\mathcal{L}^{(\text{lin})}_{NMG} = \tfrac{1}{2} h^{\mu\nu} G^{(\text{lin})}_{\mu\nu} + \tfrac{1}{m^2} G^{\mu\nu}_{(\text{lin})} S^{(\text{lin})}_{\mu\nu}$$

➽ This is quadratic approximation to Lagrangian of "new massive gravity" [BHT]

$$\mathcal{L}_{NMG} = -\sqrt{|g|}R + \tfrac{1}{m^2}\sqrt{|g|}\,K\,, \qquad K = R^{\mu\nu}R_{\mu\nu} - \tfrac{3}{8}R^2$$

Note "**wrong sign**" for Einstein-Hilbert term.

➽ Linearized NMG is *on-shell* equivalent to FP **by construction** but is it *off-shell* equivalent? In other words, *is linearized NMG unitary*, or will we find that one spin 2 mode is a ghost (like EP for spin 1)?

Auxiliary field formulation of NMG

Consider the Lagrangian [BHT]

$$\mathcal{L} = \sqrt{|g|}\,\left[-R + f^{\mu\nu}G_{\mu\nu} - \tfrac{1}{4}m^2\left(f^{\mu\nu}f_{\mu\nu} - f^2\right)\right]\,, \qquad f = g^{\mu\nu}f_{\mu\nu}$$

The symmetric tensor field $f_{\mu\nu}$ is auxiliary: its equation of motion is

$$f_{\mu\nu} = \tfrac{2}{m^2}S_{\mu\nu}\,, \qquad S_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}R$$

The tensor $S_{\mu\nu}$ is the 3D Schouten tensor (gauge-potential for conformal boosts).

- If we eliminate $f_{\mu\nu}$ by its algebraic equation of motion we recover NMG
- Define $\bar{h}_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} - f_{\mu\nu}$, and linearize in $(\bar{h}_{\mu\nu}, f_{\mu\nu})$. Quadratic Lagrangian is

$$\mathcal{L}_{quad} = -\mathcal{L}^{(\text{lin})}_{EH}(\bar{h}) + \mathcal{L}_{FP}(f)$$

Note the "wrong sign" for the EH term, **but no modes are propagated by this term**. Hence equivalence to FP [BHT, Nakasone & Oda, '09]

Aside: Scalar Massive Gravity

Consider the Lagrangian

$$\mathcal{L}_{SMG} = \sqrt{|g|}\,\left[R - \frac{1}{2m^2}R^2\right]$$

This is equivalent, in any dimension, to a scalar field coupled to Einstein gravity, and hence is unitary [Bicknell, '74]

- Proof starts with equivalent Lagrangian involving auxilary *scalar* field f:

$$\mathcal{L} = \sqrt{|g|}\,\left[R - fR + \tfrac{1}{2}m^2 f^2\right]$$

- Proceeding in 3D, we define new metric, g' and scalar field ϕ by

$$g'_{\mu\nu} = (1+f)^2\, g_{\mu\nu}\,, \qquad e^{-\phi} = 1+f$$

to get to new Lagrangian

$$\mathcal{L} = \sqrt{|g'|}\,\left[R' - 2\,(\partial\phi)^2 - 2m^2\phi^2 + \mathcal{O}(\phi^3)\right]$$

Canonical structure of NMG

- Metric perturbation $h_{\mu\nu}$ has only 3 degrees of freedom because of gauge invariance

- $\mu \to (0, i = 1, 2)$. Choose gauge $\partial_i h_{i\mu} = 0$

- We may write components of $h_{\mu\nu}$ in this gauge as

$$h_{ij} = -\varepsilon^{ik}\varepsilon^{jl}\frac{\partial_k\partial_\ell}{\nabla^2}\varphi\,, \qquad h_{0i} = -\varepsilon^{ij}\frac{1}{\nabla^2}\partial_j\xi\,, \qquad h_{00} = \frac{1}{\nabla^2}(N + \Box\varphi)$$

The three functions (φ, ξ, N) are independent degrees of freedom. **We allow non-locality in space but not in time**.

- Substitiute into action. Define $\zeta = \xi/m$. N is auxiliary, eliminate to get [Deser, '09]

$$\mathcal{L} = \frac{1}{2}\left[\varphi\Box\varphi + \zeta\Box\zeta\right] - \frac{1}{2}m^2\left[\varphi^2 + \zeta^2\right]$$

No higher time derivatives!. But maintaining space locality yields action 4th order in space derivatives (**cf Hořava gravity**).

- Because of space non-locality relative to linearized NMG, cannot interpret (φ, ζ) as scalars. In fact, they have helicity ± 2.

Renormalizability?

Consider for $D = 3, 4$ and $\sigma = \pm 1$

$$S[g] = \int d^D x \sqrt{|g|} \left[\sigma R + a R^{\mu\nu} R_{\mu\nu} + b R^2\right]$$

➽ $D = 4$. By Bicknell's theorem, model is unitary but non-renormalizable if $a = 0$.

➽ $D = 4$ non-unitary if $a \neq 0$, but then **renormalizable** provided that $a \neq 3b$ [Stelle, '77]

➽ $D = 3$ and $\sigma = 1$. Unitary but non-renormalizable for $a = 0$. This is SMG

➽ $D = 3$ and $\sigma = -1$. Unitary for $3a - 8b = 0$. This is NMG. As $D = 4$ theory is renormalizable, and $a(a - 3b) \neq 0$, expect super-renormalizability. Has been checked [Oda, '09] but questions remain [Deser, '09]

Generalized Massive Gravity

Starting from generalized FP equations and solving differential subsidiary condition, we get linearized GMG equations

$$[\mathcal{P}(m_+)\mathcal{P}(-m_-)]_\mu{}^\rho G^{(\text{lin})}_{\rho\nu} = 0\,, \qquad R^{(\text{lin})} = 0$$

These equations follow from Lagrangian

$$\mathcal{L}^{(\text{lin})}_{GMG} = \tfrac{1}{2}h^{\mu\nu}G^{(\text{lin})}_{\mu\nu} + \tfrac{1}{2\mu}h^{\mu\nu}C^{(\text{lin})}_{\mu\nu} + \tfrac{1}{m^2}G^{\mu\nu}_{(\text{lin})}S^{(\text{lin})}_{\mu\nu}$$

- Unitary. 'Canonical' analysis shows that there are two physical modes with squared masses $m^2_\pm$ [ABdRHST]. These are helicity +2 and helicity −2 modes

- Quadratic approximation to GMG Lagrangian [BHT]

$$\mathcal{L}_{GMG} = -\sqrt{|g|}R + \tfrac{1}{\mu}\mathcal{L}_{LCS} + \tfrac{1}{m^2}\sqrt{|g|}\,K$$

Note "wrong sign" for Einstein-Hilbert term.

- Recover TMG in limit $m^2 \to \infty$. Recover NMG in limit $|\mu| \to \infty$.

$\mathcal{N} = 2$ *Unification*

Spin 1 and Spin 2 models unified in $\mathcal{N} = 2$ Sugra [ABdRHST]. Bosonic fields are metric and 'auxiliary' vector B_μ.

- $\mathcal{L}_{EH} \to \mathcal{L}_{EH} + B^2$
- $\mathcal{L}_{LCS} \to \mathcal{L}_{LCS} + \varepsilon^{\mu\nu\rho} B_\mu \partial_\nu B_\rho$
- $K \to K + \tilde{G}^2(B)$

Hence

- GMG $\to$ GMG + GP

Note that "wrong-sign" for EH term is needed to get right sign for B mass term, and unitarity of GP fixes sign of K.

- What do we get in Spin 2 sector if we add ECS term to spin 1 sector? We get ELCS term [BHT]

$$\mathcal{L}_{ELCS} = G^{\mu\nu} C_{\mu\nu} \equiv \varepsilon^{\mu\nu\rho} S_\mu{}^\alpha \partial_\nu S_{\rho\alpha} + \ldots$$

Leads to ghosts, just like ECS for spin 1, as required by $\mathcal{N} = 2$ susy.

Cosmological models

- Add cosmological term [BHT]:

$$\mathcal{L}_{CNMG} = \sqrt{|g|}\left[-2\lambda m^2 + \sigma R + \frac{1}{m^2}K\right] + \frac{1}{\mu}\mathcal{L}_{LCS}$$

The parameter λ is dimensionless. Either sign of EH term ($\sigma = 1$ is "right sign"). Allow $m^2 < 0$.

- Look for maximally symmetric vacua, i.e.

$$G_{\mu\nu} = -\Lambda g_{\mu\nu}$$

for (cosmological) constant Λ ($\Lambda < 0 \Rightarrow$ AdS). Find that

$$\Lambda^2 + 4m^2\sigma\Lambda - 4\lambda m^4 = 0 \Rightarrow \Lambda = -2m^2\left[\sigma \pm \sqrt{1+\lambda}\right]$$

- $\lambda = -1$: 'partially massless' [Deser & Nepomechie, '84]

- Bulk unitarity bound $\sigma(\lambda - 3) \geq 0$ [BHT]

- Saturation of bulk unitarity bound ($\lambda = 3$) $\Rightarrow$ massive graviton in adS bulk disappears! – replaced by massive photon

adS_3/CFT_2

In adS we expect boundary CFT [Brown & Henneaux, Maldacena,. . .]. Central charge can be computed by various methods [Henningson & Skenderis, Saida & Soda, Kraus & Larsen, ... In our case [Liu&Sun, BHT]

$$c_{\pm} = \frac{3\ell}{2G_3}\left(\sigma + \frac{1}{2\ell^2 m^2} + \frac{1}{\mu\ell}\right)$$

where $\ell = 1\sqrt{-\Lambda}$ is adS radius and G_3 the 3D Newton constant.

- e.g. $\mu \to \infty$ limit: $c_{\pm} = c_{CNMG} = \frac{3\ell}{2m^2 G_3}\left(|\Lambda| - 2m^2\sigma\right)$

- Central charge can vanish when $\sigma m^2 > 0$. This happens at $\lambda = 3$, i.e. when bulk gravitons are absent!

- Unitarity in bulk $\Leftrightarrow$ negative c.c. of boundary CFT!

- This is essentially the same problem as in cosmological TMG [Carlip, Deser, Waldron & Wise] – possibly solved by "chiral gravity" [Li, Song & Strominger] (and possibly not [Skenderis, Taylor & van Rees].

- Vacuum structure quite different for $\mathcal{N} = 1$ 3D massive sugra [ABdRHST]. Unitarity in adS vacua not yet analysed.

Outlook

- ➽ Enlarged class of generally covariant unitary theories for massive spin 2 in 3D; includes interacting extension of Fierz-Pauli.

- ➽ Higher derivative but higher *time* derivatives cancel in linearized theory; need full ADM formalism to check that all is still OK in the full theory–to be done.

- ➽ Renormalizability suggested by 4D results of Stelle, and there are claims for TMG [Deser & Zhang] and NMG [Oda]. More work needed.

- ➽ Cosmological version with adS vacuum suffers same problem as TMG but sugra may help–we shall see.

- ➽ $\mathcal{N} = 1$ sugra extensions constructed; $\mathcal{N} > 1$ is technically challenging. Expect $\mathcal{N}_{max} = 8$ for NMG, but $\mathcal{N}_{max} = 7$ for GMG and TMG. Linearized construction under way.

- ➽ Higher spin. Preliminary results suggest that massive spin s can be described by a gauge theory of order $2s$ in derivatives.