Non-commutative closed string geometry from flux compactifications

Dieter Lüst, LMU (Arnold Sommerfeld Center)
and MPI München

Non-commutative closed string geometry from flux compactifications

Dieter Lüst, LMU (Arnold Sommerfeld Center)
and MPI München

SUPERFIELDS

European Research Council

Adv. Inv. Grant no. 226455

Padova, 18. March 2011

I) Introduction

Closed string flux compactifications:

- Moduli stabilization ➡ string landscape
- AdS/CFT correspondence
- Generalized geometries
- Here: closed string non-commutative (non-associative) geometry

Non-commutative geometry and string theory (a):

Open strings:

2-dimensional D-branes with 2-form F-flux $\Rightarrow$ coordinates of open string end points become non-commutative:

$$[X_i(\tau), X_j(\tau)] = \epsilon_{ij}\Theta\,, \qquad \Theta = -\frac{2\pi i\alpha' F}{1+F^2}$$

(A. Abouelsaood, C. Callan, C. Nappi, S. Yost (1987);
J. Fröhlich, K. Gawedzki (1993); V. Schomerus (1999); ....)

Non-commutative geometry and string theory (a):

Open strings:

2-dimensional D-branes with 2-form F-flux $\Rightarrow$

coordinates of open string end points become non-commutative:

$$[X_i(\tau), X_j(\tau)] = \epsilon_{ij}\Theta\,, \qquad \Theta = -\frac{2\pi i\alpha' F}{1+F^2}$$

constant

(A. Abouelsaood, C. Callan, C. Nappi, S. Yost (1987); J. Fröhlich, K. Gawedzki (1993); V. Schomerus (1999); ....)

Non-commutative geometry and string theory (a):

Open strings:

2-dimensional D-branes with 2-form F-flux $\Rightarrow$

coordinates of open string end points become non-commutative:

$$[X_i(\tau), X_j(\tau)] = \epsilon_{ij}\Theta\,, \qquad \Theta = -\frac{2\pi i\alpha' F}{1+F^2}$$

constant

(A. Abouelsaood, C. Callan, C. Nappi, S. Yost (1987); J. Fröhlich, K. Gawedzki (1993); V. Schomerus (1999); ....)

➢ Non-commutative gauge theories.

(N. Seiberg, E. Witten (1999); J. Madore, S. Schraml, P. Schupp, J. Wess (2000); ....)

Moyal-Weyl $\star$- product:

$$f_1(x) \star f_2(x) \star \ldots \star f_N(x) :=$$

$$\exp\left[i \sum\nolimits_{m<n} \Theta^{ab}\, \partial_a^{x_m}\, \partial_b^{x_n}\right]\, f_1(x_1)\, f_2(x_2) \ldots f_N(x_N)\Big|_{x_1=\ldots=x_N=x}$$

$$S \simeq \int d^n x\, \mathrm{Tr} \hat{F}_{ab} \star \hat{F}^{ab}$$

Non-commutative geometry and string theory (b):

Closed strings:

3-dimensional backgrounds with 3-form flux $\Rightarrow$

Non-commutative geometry and string theory (b):

Closed strings:

3-dimensional backgrounds with 3-form flux $\Rightarrow$

we will show that coordinates of closed strings become non-commutative:

(D.L., arXiv:1010.1361)

$$[X_i(\tau,\sigma), X_j(\tau,\sigma)] \simeq F^{(3)}_{ijk}\ p^k$$

Non-commutative geometry and string theory (b):

Closed strings:

3-dimensional backgrounds with 3-form flux $\Rightarrow$

we will show that coordinates of closed strings become non-commutative:

(D.L., arXiv:1010.1361)

$$[X_i(\tau,\sigma), X_j(\tau,\sigma)] \simeq F^{(3)}_{ijk}\ p^k$$

operator

Non-commutative geometry and string theory (b):

Closed strings:

3-dimensional backgrounds with 3-form flux $\Rightarrow$

we will show that coordinates of closed strings become non-commutative:

(D.L., arXiv:1010.1361)

$$[X_i(\tau,\sigma), X_j(\tau,\sigma)] \simeq F^{(3)}_{ijk}\; p^k$$

operator

and even non-associative:

(R. Blumenhagen, E. Plauschinn, arXiv:1010.1263)

$$[[X_i(\tau,\sigma), X_j(\tau,\sigma)], X_k(\tau,\sigma)] + \text{perm.} \simeq F^{(3)}_{ijk}$$

➢ Non-commutative/non-associative gravity?

Non-commutative geometry and string theory (b):

Closed strings:

3-dimensional backgrounds with 3-form flux $\Rightarrow$

we will show that coordinates of closed strings become non-commutative:

(D.L., arXiv:1010.1361)

$$[X_i(\tau,\sigma), X_j(\tau,\sigma)] \simeq F^{(3)}_{ijk}\ p^k$$

operator

and even non-associative:

(R. Blumenhagen, E. Plauschinn, arXiv:1010.1263)

$$[[X_i(\tau,\sigma), X_j(\tau,\sigma)], X_k(\tau,\sigma)] + \text{perm.} \simeq F^{(3)}_{ijk}$$

constant

➢ Non-commutative/non-associative gravity?

Outline:

II) T-duality

III) Non-commutative geometry

IV) Algebraic structure and
new uncertainty relations

V) Outlook (non-associative gravity)

II) T-duality

How does a closed string see geometry?

Consider compactification on a circle with radius R:

$$X(\tau,\sigma) = X_L(\tau+\sigma) + X_R(\tau-\sigma)$$

$$X_L(\tau+\sigma) = \frac{x}{2} + p_L(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-in(\tau+\sigma)},$$

$$X_R(\tau-\sigma) = \frac{x}{2} + p_R(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n e^{-in(\tau-\sigma)}$$

$$p_L = \frac{1}{2}\left(\frac{M}{R} + (\alpha')^{-1}NR\right), \qquad p = p_L + p_R = \frac{M}{R}$$

(KK momenta)

$$p_R = \frac{1}{2}\left(\frac{M}{R} - (\alpha')^{-1}NR\right) \qquad \tilde{p} = p_L - p_R = (\alpha')^{-1}NR$$

(dual momenta - winding modes)

T-duality: $T: \quad R \longleftrightarrow \frac{\alpha'}{R}, \quad M \longleftrightarrow N$

$$T: \quad p \longleftrightarrow \tilde{p}\,, \quad p_L \longleftrightarrow p_L\,,, \quad p_R \longleftrightarrow -p_R\,.$$

- Dual space coordinates: $\tilde{X}(\tau,\sigma) = X_L - X_R$

$(X, \tilde{X})$: Doubled geometry: (O. Hohm, C. Hull, B. Zwiebach (2009/10))

T-duality is part of diffeomorphism group.

$$T: \quad X \longleftrightarrow \tilde{X}\,, \quad X_L \longleftrightarrow X_L\,, \quad X_R \longleftrightarrow -X_R$$

- Shortest possible radius: $R \geq R_c = \sqrt{\alpha'}$

Compactification on a 2-dimensional torus:

Background: $R_1, \; R_2, \; e^{i\alpha}, \; B$

2 complex background parameters:

$$\tau = \frac{e_2}{e_1} = \frac{R_2}{R_1} e^{i\alpha},$$
$$\rho = B + i R_1 R_2 \sin\alpha \,.$$

T-duality transformations:

- $SL(2,\mathbb{Z})_\tau : \quad \tau \rightarrow \dfrac{a\tau + b}{c\tau + d}$
- $SL(2,\mathbb{Z})_\rho : \quad \rho \rightarrow \dfrac{a\rho + b}{c\rho + d}$

They act as shifts/rotations on doubled coordinates.

- T-duality in x_1 $\Leftrightarrow$ Mirror symmetry:

$$\tau \leftrightarrow \rho \iff B \leftrightarrow \Re\tau$$

Three-dimensional backgrounds ⇒ twisted 3-tori:

(A. Dabholkar, C. Hull (2003) ; S. Hellerman, J. McGreevy, B. Williams (2004); J. Derendinger, C. Kounnas, P. Petropoulos, F. Zwirner (2004); J. Shelton, W. Taylor, B. Wecht (2005); G. Dall'Agata, S. Ferrara (2005)...)

Fibrations: 2-dim. torus that varies over a circle:

$$T^2_{x^1,x^2} \hookrightarrow M^3 \hookrightarrow S^1_{x^3}$$

The fibration is specified by its monodromy properties.

Three-dimensional backgrounds ⇒ twisted 3-tori:

(A. Dabholkar, C. Hull (2003) ; S. Hellerman, J. McGreevy, B. Williams (2004); J. Derendinger, C. Kounnas, P. Petropoulos, F. Zwirner (2004); J. Shelton, W. Taylor, B. Wecht (2005); G. Dall'Agata, S. Ferrara (2005)...)

Fibrations: 2-dim. torus that varies over a circle:

$$T^2_{x^1,x^2} \hookrightarrow M^3 \hookrightarrow S^1_{x^3}$$

The fibration is specified by its monodromy properties.

Two (T-dual) cases:

(i) Geometric spaces (manifolds)

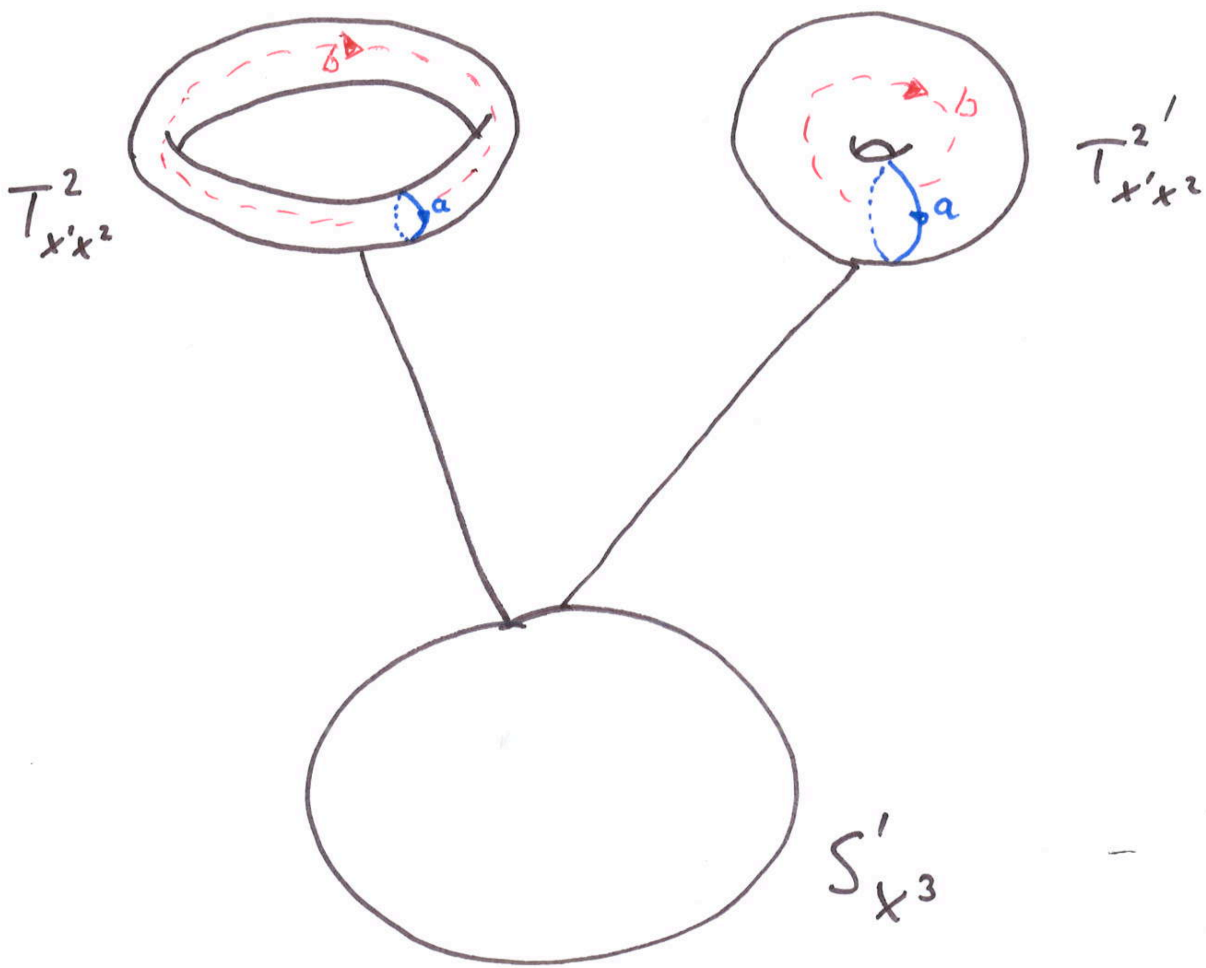
$$x^3 \to x^3 + 2\pi \;\Rightarrow\; \tau(x^3 + 2\pi) = \frac{a\tau(x^3) + b}{c\tau(x^3) + d}$$

Thr ... ori:

$$\tau(x^3+2\pi) = -1/\tau(x^3)$$

c ... ;
rara (2005)...)

Fibratio



The fib ... ties.

Two (T

(i)

$x^{\cdots}$ ... $\frac{\cdots) + b}{\cdots) + d}$

Three-dimensional backgrounds ⇒ twisted 3-tori:

(A. Dabholkar, C. Hull (2003) ; S. Hellerman, J. McGreevy, B. Williams (2004); J. Derendinger, C. Kounnas, P. Petropoulos, F. Zwirner (2004); J. Shelton, W. Taylor, B. Wecht (2005); G. Dall'Agata, S. Ferrara (2005)...)

Fibrations: 2-dim. torus that varies over a circle:

$$T^2_{x^1,x^2} \hookrightarrow M^3 \hookrightarrow S^1_{x^3}$$

The fibration is specified by its monodromy properties.

Two (T-dual) cases:

(i) Geometric spaces (manifolds)

$$x^3 \rightarrow x^3 + 2\pi \;\Rightarrow\; \tau(x^3 + 2\pi) = \frac{a\tau(x^3) + b}{c\tau(x^3) + d}$$

Three-dimensional backgrounds $\Rightarrow$ twisted 3-tori:

(A. Dabholkar, C. Hull (2003) ; S. Hellerman, J. McGreevy, B. Williams (2004); J. Derendinger, C. Kounnas, P. Petropoulos, F. Zwirner (2004); J. Shelton, W. Taylor, B. Wecht (2005); G. Dall'Agata, S. Ferrara (2005)...)

Fibrations: 2-dim. torus that varies over a circle:

$$T^2_{x^1,x^2} \hookrightarrow M^3 \hookrightarrow S^1_{x^3}$$

The fibration is specified by its monodromy properties.

Two (T-dual) cases:

(i) Geometric spaces (manifolds)

$$x^3 \to x^3 + 2\pi \;\Rightarrow\; \tau(x^3 + 2\pi) = \frac{a\tau(x^3) + b}{c\tau(x^3) + d}$$

(ii) Non-geometric spaces (T-folds)

$$x^3 \to x^3 + 2\pi \;\Rightarrow\; \rho(x^3 + 2\pi) = \frac{a\rho(x^3) + b}{c\rho(x^3) + d}$$

$$\rho(x^3 + 2\pi) = -1/\rho(x^3)$$

tori:

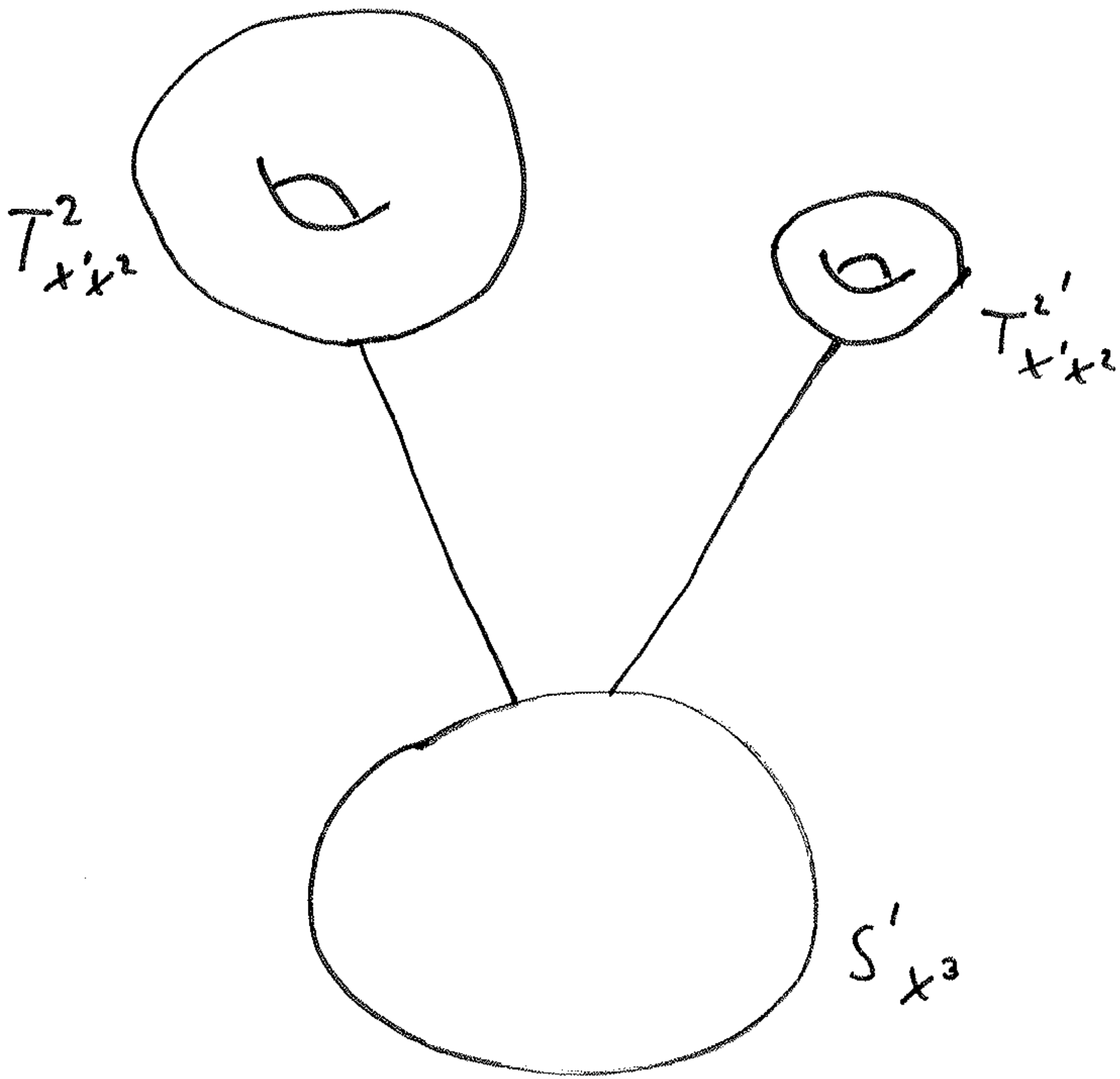
er,
rrara (2005)...)

Fib

Th

rties.

Tv



$) + b$

$) + d$

$+ b$

$+ d$

Three-dimensional backgrounds ⇒ twisted 3-tori:

(A. Dabholkar, C. Hull (2003) ; S. Hellerman, J. McGreevy, B. Williams (2004); J. Derendinger, C. Kounnas, P. Petropoulos, F. Zwirner (2004); J. Shelton, W. Taylor, B. Wecht (2005); G. Dall'Agata, S. Ferrara (2005)...)

Fibrations: 2-dim. torus that varies over a circle:

$$T^2_{x^1,x^2} \hookrightarrow M^3 \hookrightarrow S^1_{x^3}$$

The fibration is specified by its monodromy properties.

Two (T-dual) cases:

(i) Geometric spaces (manifolds)

$$x^3 \to x^3 + 2\pi \;\Rightarrow\; \tau(x^3 + 2\pi) = \frac{a\tau(x^3) + b}{c\tau(x^3) + d}$$

(ii) Non-geometric spaces (T-folds)

$$x^3 \to x^3 + 2\pi \;\Rightarrow\; \rho(x^3 + 2\pi) = \frac{a\rho(x^3) + b}{c\rho(x^3) + d}$$

Two different kind of monodromies for the fibrations:

(i) elliptic monodromies: finite order

$$SL(2,\mathbb{Z})_\tau, SL(2,\mathbb{Z})_\rho : \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

order 4 order 6

(ii) parabolic monodromies: infinite order

$$SL(2,\mathbb{Z})_\tau, SL(2,\mathbb{Z})_\rho : \quad \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$$

Both types in general contain geometric spaces as well as non-geometric backgrounds.

III) Non-commutative geometry

3.1) Open strings on D2-branes:

(i) D2-branes with gauge F-flux

Mixed D/N boundary conditions:

$$\begin{aligned} \partial_\sigma X_1 + F_{12}\,\partial_\tau X_2 &= 0, \\ \partial_\sigma X_2 - F_{12}\partial_\tau X_1 &= 0 \end{aligned}$$

$$[X_1(\tau,0),X_2(\tau,0)] = -\frac{2\pi i\alpha' F_{12}}{1+(F_{12})^2}$$

$\updownarrow$ T-duality (Seiberg-Witten map)

T-duality in X_1 :

(ii) D1-branes at angles

Boundary conditions:

$$\begin{aligned} N: \quad \partial_\sigma X_1 + F_{12}\,\partial_\sigma X_2 &= 0, \\ D: \quad \partial_\tau X_2 - F_{12}\,\partial_\tau X_1 &= 0. \end{aligned}$$

$$[X_1(\tau,0),X_2(\tau,0)] = 0$$

Geom. angle: $\nu = \dfrac{\operatorname{arccot} F_{12}}{\pi}$

Open string CFT with F-flux is exactly solvable $\Rightarrow$

shifted oscillator frequencies:

$$X_1 = x_1 - \sqrt{\alpha'} \sum_{n \in Z} \frac{\alpha_{n+\nu}}{n+\nu} e^{-i(n+\nu)\tau} \sin[(n+\nu)\sigma + \theta_1] - \sqrt{\alpha'} \sum_{m \in Z} \frac{\alpha_{m-\nu}}{m-\nu} e^{-i(m-\nu)\tau} \sin[(m-\nu)\sigma - \theta_1],$$

$$X_2 = x_2 + i\sqrt{\alpha'} \sum_{n \in Z} \frac{\alpha_{n+\nu}}{n+\nu} e^{-i(n+\nu)\tau} \sin[(n+\nu)\sigma + \theta_1] - i\sqrt{\alpha'} \sum_{m \in Z} \frac{\alpha_{m-\nu}}{m-\nu} e^{-i(m-\nu)\tau} \sin[(m-\nu)\sigma - \theta_1].$$

(A. Abouelsaood, C. Callan, C. Nappi, S. Yost (1987); C. Chu, P. Ho (1999))

$$\nu = \frac{\text{arccot}\, F_{12}}{\pi}$$

3.2) Closed strings on a 3-dim. space:

Can the closed string also see a non-commutative space?

What deformation is needed?

Yes: one needs 3-form flux: H/ω/Q/R

3.2) Closed strings on a 3-dim. space:

Can the closed string also see a non ... ive space?

NS H-flux

What deformation is needed?

Yes: one needs 3-form flux: H/ω/Q/R

3.2) Closed strings on a 3-dim. space:

Can the closed string also see a non-co space?

metric flux

What deformation is needed?

Yes: one needs 3-form flux: H/ω/Q/R

3.2) Closed strings on a 3-dim. space:

Can the closed string also see a non-comm ace?

What deformation is needed?

non-geom. flux

Yes: one needs 3-form flux: H/ω/Q/R

3.2) Closed strings on a 3-dim. space:

Can the closed string also see a non-commu e?

What deformation is needed?

R-flux

Yes: one needs 3-form flux: H/ω/Q/R

3.2) Closed strings on a 3-dim. space:

Can the closed string also see a non-commu e?

R-flux

What deformation is needed?

Yes: one needs 3-form flux: H/ω/Q/R

(i) Geometric spaces (manifolds)

$$[X^1(\tau,\sigma), X^2(\tau,\sigma)] = 0$$

3.2) Closed strings on a 3-dim. space:

Can the closed string also see a non-commu R-flux e?

What deformation is needed?

Yes: one needs 3-form flux: H/ω/Q/R

(i) Geometric spaces (manifolds)

$$[X^1(\tau,\sigma), X^2(\tau,\sigma)] = 0$$

(ii) Non-geometric spaces (T-folds) $\updownarrow$ T-duality

$$[X^1(\tau,\sigma), X^2(\tau,\sigma)] \neq 0$$

3.2) Closed strings on a 3-dim. space:

Can the closed string also see a non-commu e?

R-flux

What deformation is needed?

Yes: one needs 3-form flux: H/ω/Q/R

(i) Geometric spaces (manifolds)

$$[X^1(\tau,\sigma), X^2(\tau,\sigma)] = 0 \;\; ([X^1(\tau,\sigma), \tilde{X}^2(\tau,\sigma)] \neq 0)$$

(ii) Non-geometric spaces (T-folds) $\updownarrow$ T-duality

$$[X^1(\tau,\sigma), X^2(\tau,\sigma)] \neq 0$$

More general:

Doubled geometry: Closed string non-commutativity in $(X, \tilde{X})$-space

Problem:

- Background is non-constant.
- CFT is in general not exactly solvable

Ways to handle:

- Study SU(2) WZW model with H-flux
  (R. Blumenhagen, E. Plauschinn, arXiv:1010.1263)
- Consider sigma model perturbation theory for small H-field
  (R. Blumenhagen, A. Deser, D.L., E. Plauschinn, work in progress)
- Consider monodromy properties and the corresponding closed string boundary conditions

$\Rightarrow$ Shifted closed string mode expansion

Specific example: elliptic monodromy

C. Hull, R. Reid-Edwards (2009))

(i) Geometric space (ω -flux) $(\omega_{123} \sim \partial_{x^3} g_{x^1 x^2} \sim \partial_{x^3} \Re\tau(x^3))$

$$\tau(x^3) = \frac{(1+i)\cos(Hx^3) + \sin(Hx^3)}{\cos(Hx^3) - (1+i)\sin(Hx^3)} \quad (H \in \frac{1}{4} + \mathbb{Z})$$

Monodromy: $\tau(x^3 + 2\pi) = -1/\tau(x^3)$

S

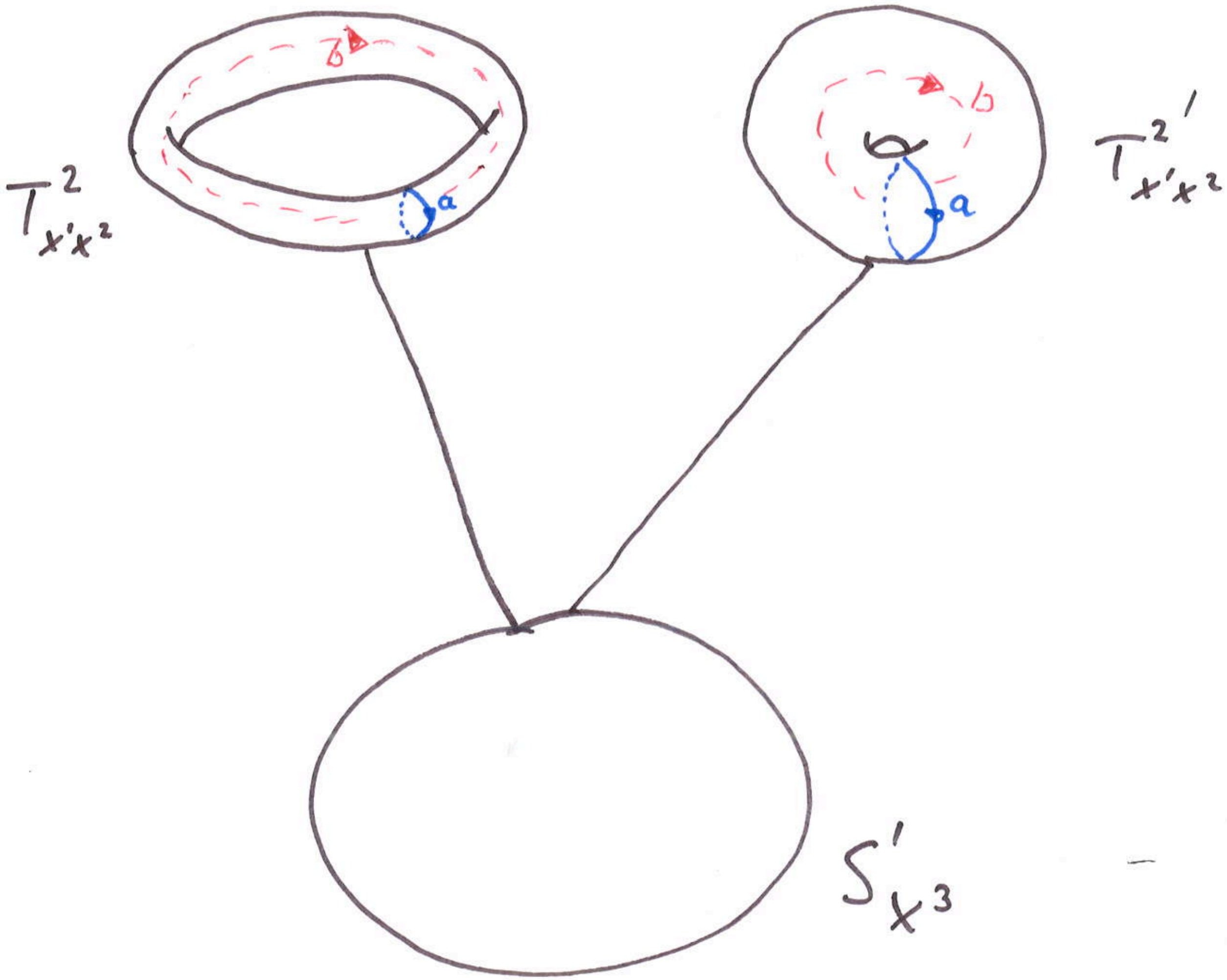
$$\tau(x^3+2\pi) = -1/\tau(x^3)$$

d-Edwards (2009))

(i) Ge

$(\omega_{123} \sim \partial_{x^3} g_{x^1 x^2} \sim \partial_{x^3} \Re\tau(x^3))$

$\frac{1}{4} + \mathbb{Z})$



Specific example: elliptic monodromy

C. Hull, R. Reid-Edwards (2009))

(i) Geometric space (ω -flux) $(\omega_{123} \sim \partial_{x^3} g_{x^1 x^2} \sim \partial_{x^3} \Re\tau(x^3))$

$$\tau(x^3) = \frac{(1+i)\cos(Hx^3) + \sin(Hx^3)}{\cos(Hx^3) - (1+i)\sin(Hx^3)} \quad (H \in \frac{1}{4} + \mathbb{Z})$$

Monodromy: $\tau(x^3 + 2\pi) = -1/\tau(x^3)$

Specific example: elliptic monodromy

C. Hull, R. Reid-Edwards (2009))

(i) Geometric space (ω -flux) $(\omega_{123} \sim \partial_{x^3} g_{x^1 x^2} \sim \partial_{x^3} \Re\tau(x^3))$

$$\tau(x^3) = \frac{(1+i)\cos(Hx^3) + \sin(Hx^3)}{\cos(Hx^3) - (1+i)\sin(Hx^3)} \quad (H \in \frac{1}{4} + \mathbb{Z})$$

Monodromy: $\tau(x^3 + 2\pi) = -1/\tau(x^3)$

This induces the following $\mathbb{Z}_4$ symmetric closed string boundary condition:

$$X^3(\tau, \sigma + 2\pi) = X^3(\tau, \sigma) + 2\pi N_3$$

winding number

$$X_L(\tau, \sigma + 2\pi) = e^{i\theta} X_L(\tau, \sigma), \quad \theta = -2\pi N_3 H,$$

$$X_R(\tau, \sigma + 2\pi) = e^{i\theta} X_R(\tau, \sigma).$$

L-R symmetric order 4 rotation

(Complex coordinates: $X_{L,R} = X^1_{L,R} + iX^2_{L,R}$)

Corresponding closed string mode expansion $\Rightarrow$

$$X_L(\tau+\sigma) = i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\frac{1}{n-\nu}\alpha_{n-\nu}e^{-i(n-\nu)(\tau+\sigma)}\,, \qquad \nu = \frac{\theta}{2\pi} = -N_3H\,,$$

$$X_R(\tau-\sigma) = i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\frac{1}{n+\nu}\tilde{\alpha}_{n+\nu}e^{-i(n+\nu)(\tau-\sigma)}$$ (shifted oscillators!)

Then one obtains:

$$[X_L(\tau,\sigma),\bar{X}_L(\tau,\sigma)] = -[X_R(\tau,\sigma),\bar{X}_R(\tau,\sigma)] = \tilde{\Theta}$$

$$\tilde{\Theta} = \alpha'\sum_{n\in\mathbb{Z}}\frac{1}{n-\nu} = -\alpha'\pi\cot(\pi N_3 H)$$

$$[X^1(\tau,\sigma),X^2(\tau,\sigma)] = [X^1_L+X^1_R,X^2_L+X^2_R] = 0$$

T-dual geometry (mirror symmetry): $\tau(x^3) \leftrightarrow \rho(x^3)$

(ii) Non-geometric space (Q-flux)

$$\rho(x^3) = \frac{(1+i)\cos(Hx^3) + \sin(Hx^3)}{\cos(Hx^3) - (1+i)\sin(Hx^3)} \quad \left(H \in \frac{1}{4} + \mathbb{Z}\right)$$

$\Rightarrow$ H-field: $H(x^3) = H\,\dfrac{10 - 12\sin(2Hx^3) - 6\cos(2Hx^3)}{(2\sin(2Hx^3) + \cos(2Hx^3) - 3)^2}$

Monodromy: $\rho(x^3 + 2\pi) = -1/\rho(x^3)$

T-

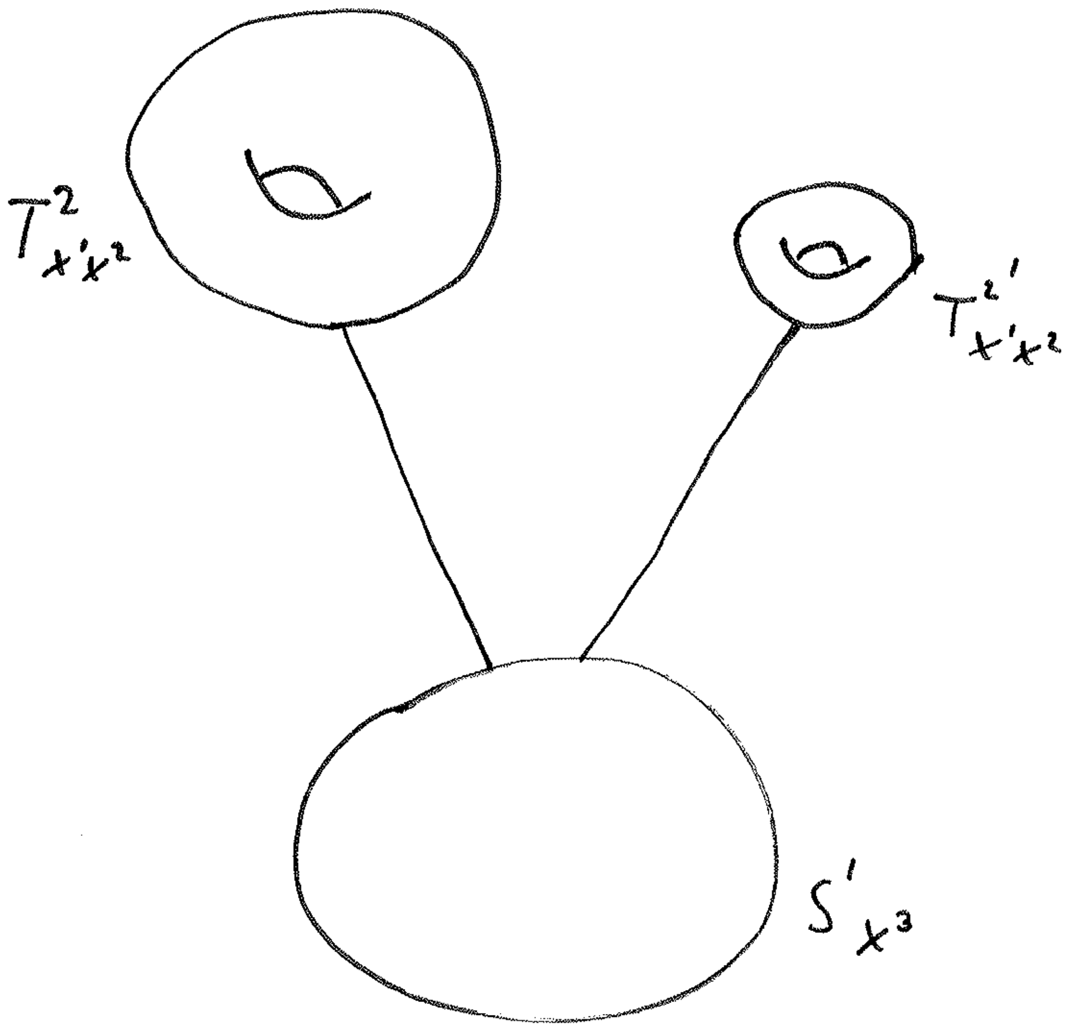
$$\rho(x^3+2\pi) = -1/\rho(x^3)$$

$\rho(x^3)$

(i

$\rho($ … $+ \mathbb{Z})$

$\frac{Hx^3)}{-3)^2}$



T-dual geometry (mirror symmetry): $\tau(x^3) \leftrightarrow \rho(x^3)$

(ii) Non-geometric space (Q-flux)

$$\rho(x^3) = \frac{(1+i)\cos(Hx^3) + \sin(Hx^3)}{\cos(Hx^3) - (1+i)\sin(Hx^3)} \quad \left(H \in \frac{1}{4} + \mathbb{Z}\right)$$

$\Rightarrow$ H-field: $H(x^3) = H\,\dfrac{10 - 12\sin(2Hx^3) - 6\cos(2Hx^3)}{(2\sin(2Hx^3) + \cos(2Hx^3) - 3)^2}$

Monodromy: $\rho(x^3 + 2\pi) = -1/\rho(x^3)$

T-dual geometry (mirror symmetry): $\tau(x^3) \leftrightarrow \rho(x^3)$

(ii) Non-geometric space (Q-flux)

$$\rho(x^3) = \frac{(1+i)\cos(Hx^3) + \sin(Hx^3)}{\cos(Hx^3) - (1+i)\sin(Hx^3)} \qquad (H \in \frac{1}{4} + \mathbb{Z})$$

$\Rightarrow$ H-field: $H(x^3) = H\dfrac{10 - 12\sin(2Hx^3) - 6\cos(2Hx^3)}{(2\sin(2Hx^3) + \cos(2Hx^3) - 3)^2}$

Monodromy: $\rho(x^3 + 2\pi) = -1/\rho(x^3)$

This induces the following $\mathbb{Z}_4$ asymmetric closed string boundary condition:

$$X^3(\tau, \sigma + 2\pi) = X^3(\tau, \sigma) + 2\pi N_3$$

$$X_L(\tau, \sigma + 2\pi) = e^{i\theta} X_L(\tau, \sigma), \quad \theta = -2\pi N_3 H,$$

$$X_R(\tau, \sigma + 2\pi) = e^{-i\theta} X_R(\tau, \sigma).$$

L-R a-symmetric order 4 rotation

Corresponding closed string mode expansion $\Rightarrow$

$$X_L(\tau+\sigma) = i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\frac{1}{n-\nu}\alpha_{n-\nu}e^{-i(n-\nu)(\tau+\sigma)}, \qquad \nu=\frac{\theta}{2\pi}=-N_3H,$$

$$X_R(\tau-\sigma) = i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\frac{1}{n+\nu}\tilde{\alpha}_{n-\nu}e^{-i(n-\nu)(\tau-\sigma)}$$

Then one finally obtains:

$$[X_L(\tau,\sigma),\bar{X}_L(\tau,\sigma)]=[X_R(\tau,\sigma),\bar{X}_R(\tau,\sigma)]=\tilde{\Theta}$$

$$[X^1(\tau,\sigma),X^2(\tau,\sigma)]=[X^1_L+X^1_R,X^2_L+X^2_R]=i\tilde{\Theta}$$

T-duality in x^3 - direction $\Rightarrow$ R-flux

Winding no. N_3 $\Longleftrightarrow$ Momentum no. M_3

$$[X^1(\tau,\sigma), X^2(\tau,\sigma)] = i\Theta$$

$$\Theta = \alpha' \sum_{n\in\mathbb{Z}} \frac{1}{n-\nu} = -\alpha' \pi \cot(\pi M_3 H)$$

Chain of T-dualities:

geom. space: $[X^1(\tau,\sigma), \tilde{X}^2(\tau,\sigma)] = i\tilde{\Theta}$

$\Updownarrow \quad T_{x^2}$

T-fold: $[X^1(\tau,\sigma), X^2(\tau,\sigma)] = i\tilde{\Theta}$

$\Updownarrow \quad T_{x^3}$

R-background: $[X^1(\tau,\sigma), X^2(\tau,\sigma)] = i\Theta$

Parabolic monodromy: (D. Andriot, M. Larfors, D.L., P. Patalong, work in progress)

Chain of four T-dual background:

$$H_{x^1x^2x^3} \xrightarrow{T_{x^1}} \omega^{x^1}_{x^2x^3} \xrightarrow{T_{x^2}} Q^{x^1x^2}_{x^3} \xrightarrow{T_{x_3}} R^{x_1x_2x_3}$$

(i) constant H-field on flat T^3 : $\quad (B_{x^1x^2} = Hx^3)$

(ii) constant metric flux ω

(iii) non-geometric Q-flux (T-fold)

(iv) R-background (not even locally a manifold)

(J. Shelton, W. Taylor, B. Wecht (2005))

H-background: $$[\tilde{\tilde{X}}^1(\tau,\sigma), \tilde{\tilde{X}}^2(\tau,\sigma)] = i\tilde{\tilde{\Theta}}$$

$\Updownarrow \quad T_{x^1}$

ω background: $$[X^1(\tau,\sigma), \tilde{\tilde{X}}^2(\tau,\sigma)] = i\tilde{\tilde{\Theta}}$$

$\Updownarrow \quad T_{x^2}$

Q-background: $$[X^1(\tau,\sigma), X^2(\tau,\sigma)] = i\tilde{\Theta}$$

$\Updownarrow \quad T_{x^3}$

R-background: $$[X^1(\tau,\sigma), X^2(\tau,\sigma)] = i\Theta$$

IV) Algebraic structure and new uncertainty relations

Act on wave functions $\Rightarrow$ replace momentum (winding)

numbers by (dual) momentum operator:

$$M_3 \equiv \sqrt{\alpha'}\, p^3 \,, \qquad N_3 \equiv \sqrt{\alpha'}\, \tilde{p}^3$$

Then one obtains the following non-commutative algebra:

$$[X^1, X^2] \simeq i l_s^3 F^{(3)}\, p^3 \quad ([X^i, X^j] \simeq i\epsilon^{ijk} F^{(3)} p^k)$$

Corresponding uncertainty relation:

$$(\Delta X^1)^2 (\Delta X^2)^2 \geq l_s^6 (F^{(3)})^2\ \langle p^3 \rangle^2$$

Use $[p^3, X^3] = -i$

$$\Longrightarrow \quad [[X^1, X^2], X^3] + \text{perm.} \simeq F^{(3)}\, l_s^3$$

Non-associative algebra!

This nicely agrees with the non-associative closed string structure found by Blumenhagen, Plauschinn in the SU(2) WZW model: arXiv:1010.1263

Finally one gets:

$$\begin{aligned} \left(\Delta[X^1, X^2]\right)^2 (\Delta X^3)^2 \quad &\simeq \quad (F^{(3)})^2\, l_s^6 (\Delta p^3)^2 (\Delta X^3)^2 \\ &\geq \quad (F^{(3)})^2\, l_s^6\,. \end{aligned}$$

V) Outlook

- Is there are non-commutative (non-associative) theory of gravity? Is there a map to commutative gravity (like SW-map for gauge theories)? (Non-commutative geometry & gravity: P. Aschieri, M. Dimitrijevic, F. Meyer, J. Wess (2005))

- What is the algebra of closed string states (functions) on this space? Is there something like a Moyal-Weyl $\star$- product?

(R. Blumenhagen, A. Deser, D.L., E. Plauschinn, work in progress)

Closed string correlation functions $\Rightarrow$

Non-associative $\triangle$- product:

$$f_1(y) \triangle f_2(y) \triangle \ldots \triangle f_N(y) :=$$

$$\exp\Big[\sum\nolimits_{m<n<r} F^{abc}\, \partial_a^{y_m}\, \partial_b^{y_n} \partial_c^{y_r}\Big]\ f_1(y_1)\, f_2(y_2) \ldots f_N(y_N)\Big|_{y_1=\ldots=y_N=y}$$